Application of Welsh-Powell Graph Coloring Algorithm in Radio Frequency Assignment

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***Abstract*—Being one of the greatest inventions of mass media in the 20th century, radio remains relevant even to this day, as evidenced by the significant proportion of Indonesia's population that continues to listen to the radio regularly. In providing good broadcasting quality, careful consideration should be made in assigning frequencies to radio stations to avoid noise resulting from frequency interference. This paper aims to study the assignment of appropriate frequencies to radio stations by applying the Welsh-Powell Coloring algorithm.**

***Keywords*—Graph, frequency assignment, coloring algorithms.**

I. Introduction

Mass communication is the process of sharing information with a large audience via mass media—that is, technology capable of sending messages to a large number of people that might be unknown to the sender [1]. Mass media is used by differrent parties for various reasons, including but not limited to advertisement, entertainment, and political propaganda.

Following the advancement of scientific inquires and engineering, the mass media has undergone a series of evolution. Johannes Gutenberg's printing press invention in the 15th century was a massive development in mass media, through which large movements—such as the Renaissance and the Protestant Reformation—was made possible. Radio is another major invention in mass media that exploded in popularity in the early 20th century which was less expensive than television and allowed for a simultaneous broadcast of the same event to a large audience [2].

Despite the emergence of newer technologies, the number of radio users is still quite significant in Indonesia. In the year 2020, 22.759 million of Indonesia's population in 10 different cities listens to the radio for approximately 2 hours per day. This number is a decline compared to the preceeding years. However, radio stations are making an effort to stay relevant by bringing conventional radio experience available digitally. This blend of conventional radio listening experience and digital media has been quite successful and it shows that the unique experience of listening to radios—lively broadcasts, up-to-date topics, imagination-arousing—is still being sought [3].

Driven by the enduring appeal of radio, the author aims to study how graph theory can be applied to radio broadcasting. Spesifically, the author will apply the Welsh-Powell algorithm to the renowned *Frequency Assignment Problem* (FAP). Through this paper, the author intends to demonstrate how coloring algorithm plays a crucial role in the frequency assignment process, ensuring optimal radio streaming quality.

II. Theoretical Basis

A. Graph

Graphs are discrete structures consisting of vertices and edges that connect these vertices. Vertices are often illustrated as dots, whereas edges connecting those vertices are illustrated with lines. Formally, a graph $G = (V, E)$ consists of V, a nonempty set of *vertices* (or *nodes*) and E, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

The branch of mathematics that studies graphs is called *graph theory*. The mathematical study of graphs has allowed it to be applied in solving real life problems. Graphs are used as a representation in modelling a wide variety of situations, including social networks, software module dependencies, airline routes, and communication networks.

B. Types of Graphs

Based on the existence of loops and parallel edges, graphs can be grouped into two categories:

1) Simple Graph
   Simple graphs are graphs in which each vertex is connected only to vertices other than itself, and the edge connecting two vertices is unique.

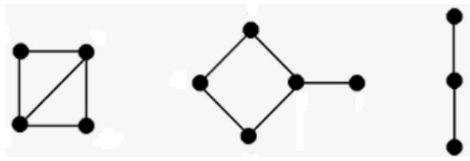

Fig. 1. Simple graphs. [5]

2) Unsimple Graph
   Unsimple graphs are graphs that contain loops or parallel edges. Graphs that contain parallel edges are called *multigraphs*, whereas graphs that contain loops with or without parallel edges are called *pseudographs*.

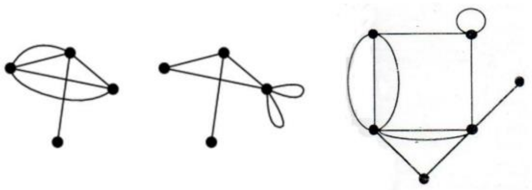

Fig. 2. Unsimple graphs. [5]

Based on the direction of edges, graphs can be grouped into two other categories:

1) Undirected Graph
Undirected graphs are graphs whose edges do not have direction. The edge connecting two vertices u and v may be written as either $e = (u, v)$ or $e = (u, v)$.

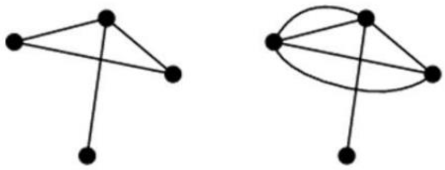

Fig. 3. Undirected graphs. [5]

2) Directed Graph
Directed graphs (or digraphs) are graphs whose edges have directions. The directed edge associated with the ordered pair (u, v) are said to *start* at u and *end* a v. Therefore, the edges $e = (u, v)$ and $e = (v, u)$ are considered distinct. Directed graphs that contain parallel edges are called *directed multigraphs*.

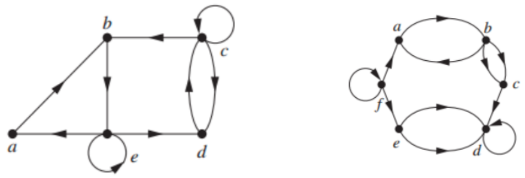

Fig. 4. Directed graphs. [5]

C. Graph Terminology

Given a graph $G = (V, E)$, vertices $u, v \in V$, and edge $e \in E$, the following terms are often employed in the discussion of graphs:

1) Adjacent
Two vertices u and v are adjacent if and only if u and v are endpoints of an edge e.
2) Incident
The edge e is said to be incident to vertices u and v if and only if u and v are adjacent.
3) Degree
For a vertex v, the degree of a vertex v (denoted by $\deg(v)$) in an undirected graph is the number of edges incident with it. A loop in a vertex adds two to its degree. In a directed graph, the in-degree of a vertex v (denoted by $\deg_{in}(v)$) is the number of edges with v as their terminal vertex. The out-degree of a vertex v is (denoted by $\deg_{out}(v)$) is the number of edges with v as their initial vertex.
4) Complete Graph
A complete graph is a simple graph in which each vertex has an edge connecting it to all other vertices. It is often represented as K_n, where n is the number of vertices in the graph.
5) Weighted Graph
A weighted graph is a graph whose edges are assigned a numeric value.

D. Graph Representations

There are several structures that can be used to represent graphs for computation. Those are:

1) Adjacency Matrix
If a graph $G = (V, E)$ has n vertices, then the adjacency matrix is an $n \times n$ matrix, where the element M_{ij} denotes the number of edges connecting vertices v_i and v_j.

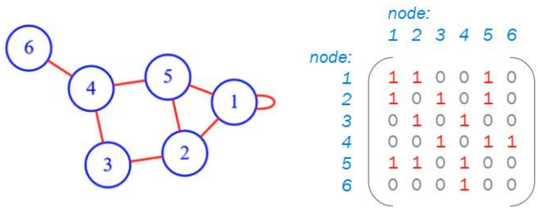

Fig. 5. Graph and its corresponding adjacency matrix. [6]

2) Adjacency List
If a graph G has n vertices, then the adjacency list is a list of n elements, where each node of the list also contains a list of all vertices adjacent to that vertex.
3) Incidence Matrix
If a graph G has n vertices and m edges, then the incidence matrix is an $n \times m$ matrix, where the element M_{ij} is 1 if the edge j is incident to vertex i and 0 if it is not.
4) Incidence List
If a graph G has n vertices, then the adjacency list is a list of n elements, where each node of the list also contains a list of all edges incident to that vertex.
5) Edge List
If a graph G has m edges, then the edge list is a $m \times 3$ matrix where for a particular row, the first column stores the edge connecting the vertices stored in the second and third columns of the respective row.

There are trade-offs between choosing one representation over the others. Adjacency matrix is a desirable representation for *dense* graphs, while adjacency list a desirable representation for *sparse* graphs. A graph is dense when it contains more than half of all possible edges, otherwise it is sparse.

E. Graph Coloring

Graph coloring is the assignment of a color to each vertex (or edge) of the graph so that no two adjacent vertices (or two different edges incident to the same vertex) are assigned the same color. The assignment of color to vertices is called *vertex coloring* and the assignment of color to edges is called *edge coloring*. In this paper, when the term graph coloring is employed, it is used to refer to vertex coloring.

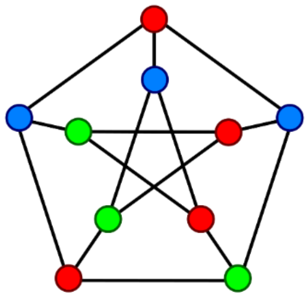

Fig. 6. Vertex coloring of a graph. [7]

In graph coloring, the *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by $\chi(G)$. The *Four-Color Theorem* states that the chromatic number of a planar graph is no greater than four.

There are many graph coloring algorithms. Finding the chromatic number of a graph is, however, an NP-hard problem. Therefore, there is no efficient algorithm in polynomial time available to solve it. Yet, there are several heuristic algorithms to color a graph which do not necessarily guarantee a coloring with the chromatic number:

1) Basic Greedy Algorithm

   A greedy algorithm repeatedly makes a locally best decision while ignoring the future effects. In coloring a graph $G = (V, E)$, the greedy algorithm approach colors the first vertex with the first color. Then, the algorithm iterates through the remaining vertices. The algorithm will color the currently picked vertex with the lowest numbered color that has not been used on any previously colored vertices adjacent to it. If all previously colors appear in the vertices adjacent to it, another color will be used.

   For $n(V) = n$ and $n(E) = m$, the worst-case scenario of this algorithm has the time complexity of $O(n^2 + m)$.

2) DSatur Algorithm

   The DSatur Algorithm is similar to the greedy algorithm. In coloring a graph $G = (V, E)$, where $n(V) = n$ and $n(E) = m$, the algorithm iterates through all vertices and color the current vertex with the lowest color $\{0, 1, 2, \ldots, n-1\}$ that is not currently assigned to the adjacent vertices. However, the vertex ordering is generated by the *degree of saturation* (DSatur), that is the number of different colors currently assigned to the adjacent vertices. The algorithm will prioritize coloring the uncolored vertex with the highest saturation degree.

   The overall complexity of this algorithm is $O(n^2)$, although optimization could be made by utilizing red-black binary tree, reducing the asymptotic complexity to $O((n + m)\log(n))$.

3) Welsh-Powell Algorithm

   The Welsh-Powell Algorithm is a modified, iterative version of the Greedy Algorithm. The Welsh-Powell Algorithm of coloring a graph $G = (V, E)$ is as follows:

   a) Find the degree of each vertex $v \in V$.
   b) Sort the vertices in order of descending degrees.
   c) Color the first vertex with the first color.
   d) Color the vertices that are not adjacent to the initial vertex with the same color.
   e) If not all vertices are colored yet, select an uncolored vertex with the largest degree and proceed to repeat step (d).

F. Radio Frequency Assignment

Radio frequency assignment is a subset of a larger problem called the *Frequency Assignment Problem* (FAP). It involves assigning the correct frequency to a particular tower so that the tower does not interfere with nearby towers. Assigning the same frequency to towers with overlapping radii would otherwise result in noise.

Graph theory is involved in the assignment problem by first making a graph model involving the radio towers. Radio towers are represented as vertex, while an edge incident to two towers means that the two towers being connected have overlapping radii. The radio frequencies would be represented as graph colors. Therefore, FAP is a coloring problem at core.

Due to the limited number of available frequencies, the assignment of frequency must be done efficiently. When two towers are located far enough that their radii are not overlapping, those two towers may be assigned with the same frequency. Here, graph coloring algorithms are utilized to assign the appropriate frequencies, ensuring minimal interference, and optimizing the overall efficiency of the frequency assignment process.

III. Discussion

A. Limitations

Assigning frequencies to a radio tower is a complex problem involving many variables, one of which is the radius of interference. However, in this paper, several limitations are set to focus the discussion specifically on the simulation of Welsh-Powell algorithm in the context of radio tower frequency assignment. The towers radii and the distance between them are set artificially.

B. Implementation of the Welsh-Powell Algorithm

The Welsh-Powell algorithm starts by finding the degree of each vertex and sorting the vertex in order of descending degrees as shown in Fig. 7 below. The list_of_vertex stores lists of three elements, storing the vertex and its corresponding degree and color. All vertices are initialized with the color 0.

```python
# Function to color a graph using the Welsh-Powell algorithm
def color_welsh(graph):
    list_of_vertex = []

    # Find the degree of a vertex
    n = len(graph)
    for i in range(n):
        temp_deg = 0
        for j in range(n):
            temp_deg += graph[i][j]
        list_of_vertex.append([i, temp_deg, 0])

    # Sort the vertices based on their degree in decreasing order
    list_of_vertex.sort(key=lambda x: x[1], reverse=True)
```

Fig. 7. Initialization of the Welsh-Powell Algorithm

After the initialization process, the algorithm continues to color the vertices by iterating through all uncolored vertices. For each uncolored vertex, it colors it with the last_color $\{1,2,\ldots\}$, and the algorithm proceeds to find all uncolored vertices and colors it with the last_color. The algorithm will return list of list of three elements, sorted ascending according to the vertex

```
last_color = 1
i = 0
while i < n:
    current_vertex = list_of_vertex[i][0]
    list_of_vertex[i][2] = last_color
    for j in range(i+1,n):
        current_vertex_two = list_of_vertex[j]
        if (not is_connected(graph,current_vertex,current_vertex_two[0])) and current_vertex_two[2] == 0:
            list_of_vertex[j][2] = last_color

    k = i
    while k < n and list_of_vertex[k][2] != 0:
        k += 1

    i = k
    last_color += 1

list_of_vertex.sort(key=lambda x:x[0])
return list_of_vertex
```

Fig. 8. Iterating through and coloring all vertices

C. Study Case I

Consider the adjacency matrix below (Fig. 9) that represents graph $G = (V, E)$, where V represent the radio towers. G is a complete graph, where each edge has a weight that represents the distance between radio towers (in km). G_{ij} represents the distance between towers $(i + 1)$ and towers $(j + 1)$.

$$G = \begin{pmatrix} 0 & 90 & 85 & 90 & 105 & 110 \\ 90 & 0 & 77 & 111 & 83 & 64 \\ 85 & 77 & 0 & 120 & 98 & 125 \\ 90 & 111 & 120 & 0 & 50 & 63 \\ 105 & 83 & 98 & 50 & 0 & 150 \\ 110 & 64 & 125 & 63 & 150 & 0 \end{pmatrix}$$

Fig. 9. Adjacency matrix of G

Let the radius of interference for this case be 100 km. The weighted graph will now be transformed into an unweighted graph $G' = (V', E')$ of 0s and 1s. For every edge with a weight greater than the radius of 100 km, the value will be 0, otherwise it will be 1. The graph is now transformed to the following:

$$G' = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 9. Adjacency matrix of G'

After the transformed graph is obtained, the graph will be colored with the Welsh-Powell Algorithm. The algorithm will first seek to find the degree of each vertex and sort the vertices in descending order of degrees.

TABLE I
VERTICES SORTED IN DESCENDING DEGREES

Vertex	Degree	Color
2	4	0
1	3	0
3	3	0
4	3	0
5	3	0
6	2	0

In the first iteration, the algorithm will color vertex 2 with the first color, and all vertices that are not adjacent with 2 the same color, resulting in the following:

TABLE II
VERTICES COLORS AFTER THE FIRST ITERATION

Vertex	Degree	Color
2	4	1
1	3	0
3	3	0
4	3	1
5	3	0
6	2	0

After the first iteration, vertices 2 and 4 are colored. The algorithm now seeks to color the uncolored vertex with the highest degree which happens to be the vertex 1. It will color vertex 1 and all vertices not adjacent with 1 with the next color.

TABLE III
VERTICES COLORS AFTER THE SECOND ITERATION

Vertex	Degree	Color
2	4	1
1	3	2
3	3	0
4	3	1
5	3	2
6	2	2

After the second iteration, the algorithm would still need to color vertex 3. This would result in the fully colored graph as shown in Table IV.

TABLE IV
VERTICES COLORS AFTER THE THIRD ITERATION

Vertex	Degree	Color
2	4	1
1	3	2
3	3	3
4	3	1
5	3	2
6	2	2

It is obvious that this coloring of the graph G is indeed a coloring with the chromatic number $\chi(G) = 3$ since the highest vertex degree in this graph is 4.

After the coloring process, the towers represented by the vertices will be assigned the frequency. The base frequency in this study case is taken to be 85 Hz and the increment to be-0.4 Hz. Therefore, the frequency assignment for this case is as shown in Fig. 10.

```
--------------RADIO FREQUENCY ASSIGNMENT--------------

The assigned frequency for each tower is:
Tower 1:        85.4 Hz
Tower 2:        85.0 Hz
Tower 3:        85.8 Hz
Tower 4:        85.0 Hz
Tower 5:        85.4 Hz
Tower 6:        85.4 Hz
```

Fig. 10. Assigned frequencies to the radio towers in case I

D. Study Case II

Consider the same graph G with the adjacency matrix in Fig. 9. If changes are made to the radius such that the radius of interference is now 50 km, the result of the program would be as shown in Fig. 11.

```
--------------RADIO FREQUENCY ASSIGNMENT--------------

The assigned frequency for each tower is:
Tower 1:        85.0 Hz
Tower 2:        85.0 Hz
Tower 3:        85.0 Hz
Tower 4:        85.0 Hz
Tower 5:        85.4 Hz
Tower 6:        85.0 Hz
```

Fig. 11. Assigned frequencies to the radio towers in case II

All towers are assigned with the same frequency except tower 5. This is as a result of all entries of the adjacency matrix after removing the weight being zero except for the edge $(4,5)$ which has the weight 50.

E. Study Case III

Consider the same graph G with the adjacency matrix in Fig. 0 but with the radius of interference 150 km. The result of the program would be as shown in Fig. 12.

```
--------------RADIO FREQUENCY ASSIGNMENT--------------

The assigned frequency for each tower is:
Tower 1:        85.0 Hz
Tower 2:        85.4 Hz
Tower 3:        85.8 Hz
Tower 4:        86.2 Hz
Tower 5:        86.6 Hz
Tower 6:        87.0 Hz
```

Fig. 12. Assigned frequencies to the radio towers in case III

All towers in this case are assigned with a different frequency. This would mean that the graph is a complete graph, where each vertex is adjacent to all vertices except for itself.

IV. Conclusion

The Welsh-Powell algorithm could be applied in assigning frequencies to radio towers. This algorithm can produce a relatively efficient coloring to a graph. Although no guarantee is made that the algorithm will produce a coloring with the chromatic number $\chi(G)$, the study case being conducted has shown the possibility of such a coloring using the aforementioned algorithm.

By collaborating with engineers from the telecommunication field and incorporating actual data, the Welsh-Powell algorithm will be a very valuable tool in this context.

V. Appendix

The implementation of the Welsh-Powell algorithm can be found in the following repository: https://github.com/julianchandras/Welsh-Powell-Frequency-Assignment

VI. Acknowledgment

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PERNYATAAN

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

Bandung, 11 Desember 2023

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