Hierarchical Structure Modeling with Combinatorial Approach and Graph Theory in Modular Origami

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***Abstract*—Modular origami has a unique pattern for each combination of *n*-vertices it possesses. However, resolving the diverse geometric configurations from a collection of origami units is challenging, especially when the modular origami have a variety of arrangement methods. This paper will determine the required amount of unit origami to make modular origami with *n*-vertices. In addition, we will analyze the problem of how the process of assembling an origami model consisting of similar pieces can be solved using graph theory. So, we have various options to create complete modular origami masterpieces.**

***Keywords*—Combinatorics, Graph, Modular Origami, Origami Structure**

I. Introduction

Origami is a form of amusement and art that involves folding paper. In origami, a sheet of paper can be folded to form a variety of shapes. One particular focus of this research lies in examining the quantitative aspects of modular origami assembly. The analysis aims to determine the number of origami units needed to create a modular origami work with a certain number of vertices or nodes (n). By exploring the mathematical relationships involved in this assembly, we aim to contribute valuable insights into optimizing the design and construction of modular origami structures.

This paper also discusses how to arrange an origami unit so that it is complete, using Hamilton's concept. This is because if we arrange it in a non-comprehensive way, then there is a possibility that the modular origami arrangement will be successful. By establishing clear conditions under which the assembly process will not fail, this research aims to contribute not only to the theoretical understanding of modular origami but also to practical applications, guiding the design and construction of reconfigurable structures with real-world implications.

Furthermore, this paper explores the arrangement of origami units using Hamilton's concept to ensure completeness. This is crucial because a non-comprehensive arrangement may lead to a failed modular origami assembly. By establishing clear conditions under which the assembly process will not fail, this research contributes not only to the theoretical understanding of modular origami but also to practical applications. It guides the design and construction of reconfigurable structures with real-world implications.

II. Graph Theory

A. Definition and Concept

In discrete mathematics, graphs are defined as ordered pairs consisting of two components: vertices or nodes and edges.

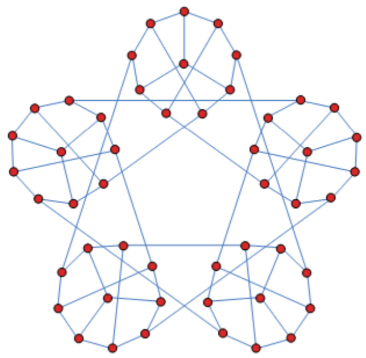

Fig 2.1. Graph Illustration
source: https://informatika.stei.itb.ac.id/~rinaldi.munir/

Denoted as $G = (V, E)$ a graph has V representing a non-empty set of vertices or nodes and E representing a set of edges connecting pairs of vertices, which could be empty. The notation for V is

$$V = \{V_1, V_2, V_3, ..., V_n\},$$

where V_i represents a $vertex$ with the index i. Similarly, the notation for E is

$$E = \{e_1, e_2, e_3, ..., e_n\},$$

where e_i represents an edge with the index i. The connection between an edge and a pair of vertices is expressed as

$$e = (V_i, V_j),$$

where e is an $edge$ linking vertices with indices i and j.

B. Type of Graph

Graphs are categorized into two types, simple graphs and unsimple graphs, based on the existence of multiple edges or loop. A simple graph is one that lacks rings or double side. Unsimple is characterized by the presence of double sides or rings.

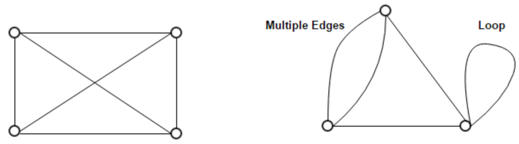

Fig 2.2. (a) Simple Graph (b) Unsimple Graph
source: https://www.javatpoint.com/

Additionally, the classification of a graph is determined by the orientation of its edges, resulting in two types: undirected graphs and directional graphs.

1. Undirected Graph
   An undirected graph lacks directional orientation in its edges.

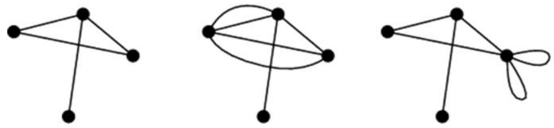

Fig 2.3. Undirected Graph
source: https://informatika.stei.itb.ac.id/~rinaldi.munir/

2. Directed Graph
   Directed graph has the type of edges with arrow that connects nodes.There are graphs that only have one direction, there are also those that have 2 directions

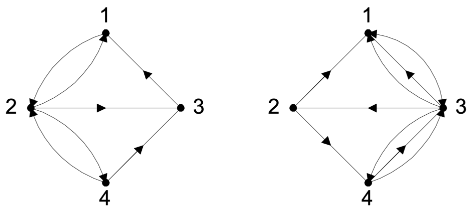

Fig 2.4. (a) Directed Graph, (b) Directed Double-Graph
source: https://informatika.stei.itb.ac.id/~rinaldi.munir/

C. Terminologies of Graph

1. Degree
   The degree of a vertex is an integer indicating the number of edges connected to the vertex. which notated as d(v). As the figure shown below, for G1, the degrees for node 1 and node 4 has the same value because it has same amount of edges incidence. Thus, d(1) = d(4) = 2.
2. Path
   A path is a sequence of edges, where the target vertex of the first edge is the source vertex of the next edge.
3. Null Graph
   A Null graph is a graph that contains no edges.Circuit
   A circuit is a path which ends at the starting vertex.
4. Dual-Graph
   The dual of graph G is the graph formed by considering the faces of G as vertices and the edges of G between two faces as the edges connecting the vertices represented by those two faces.
5. Adjacence
   Two vertices are called adjacent if they are directly connected. Based on graph G_1, vertex 1 is adjacent to vertex 2 and 3, while vertex 1 is not adjacent to vertex 4.

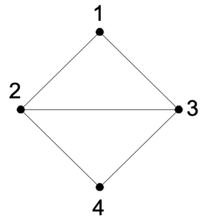

Fig 2.5. Graph with Adjacence
source: https://informatika.stei.itb.ac.id/

6. Incidency
   An edge and a vertex that is connected directly are called incidents. For any edge notated by e = (V_j, V_k) there are vertex V_j and node V_k which have incidence with the edge e. Based on figure G_1 below, it is known that edge(2,3) is incident with vertex 2 and vertex 3, edge(2,4) is incident with vertex 2 and vertex 4, while edge(1,2) has no incidence with node 4.

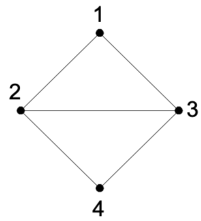

Fig 2.6. Graph with Incidency
source: https://informatika.stei.itb.ac.id/

7. Isolated Vertex
   A vertex is called isolated if it has no edge connected to it. This graph as a vertex 5 that called isolated vertex.

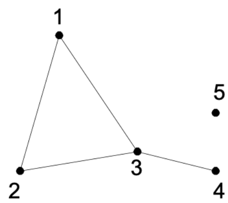

Fig 2.7. Graph with Isolated Vertex

source: https://informatika.stei.itb.ac.id/~rinaldi.munir/Matdis/2023-2024/19-Graf-Bagian1-2023.pdf

8. Subgraph
   A graph $G_1 = (V_1, E_1)$ is said to be a subgraph of graph $G = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$. A graph $G_2 = (V_2, E_2)$ is said to be a subgraph of graph G_1 if $V_2 \subseteq V$ and $E_2 = E - E_1$. The component of a graph is the maximum amount of connected subgraph in a graph G.
9. Multigraph
   A multigraph is a graph that contains parallel edges
10. Pseudograph
    A pseudograph is a graph that contains at least one edge connecting one vertex to itself (loop).
11. Spanning Subgraph
    A spanning subgraph is a subgraph which has every vertex of the graph.
12. Cut-Set
    A cut-set of a connected graph *G* is a set of edges that if discarded, results in graph *G* being disconnected, thus a cut-set always results in 2 components.

III. Origami

Origami, art of folding objects out of paper to create both two-dimensional and three-dimensional subjects. The word "origami," derived from the Japanese words "oru" meaning "to fold" and "kami" signifying "paper," has evolved into the universal descriptor for this art form. Although some European historians feel it places undue weight on the Japanese origins of an art that may well have developed independently around the world.

A. Modular Origami

Fig 3.1. Some Types of Modular Origami
source: https://www.polypompholyx.com

Modular Origami is a type of origami consisting of making models from several pieces of paper folded in the same way. This approach uniquely utilizes folds as the sole means to connect components, omitting the need for glue, tape, or string. The magic happens when these individual modules are joined together, forming a larger complex origami structure that can be quite beautiful. In essence, modular origami employs folded paper pieces as "building blocks," constructing expansive and often symmetrical formations. This creative process allows for the combination of multiple units, each folded from a single piece of paper, resulting in the crafting of more intricate and elaborate forms.

B. Sonobe Modular Origami

The Sonobe is a simple example unit from modular origami that is both easy to fold and compatible for constructing a large variety of models. The Sonobe system, named after Mitsunobu Sonobe of Japan, is comprised of models made from different numbers of the basic module, creased in different ways, as well as a module with properties related to those of the basic module. The basic module creased in different ways, along with a module exhibiting properties related to the basic unit. Each unit is folded from a square sheet of paper, with only one face visible in the finished module; designers have ingeniously created ornamental variants exposing both sides of the paper.

Fig 3.2. 90, 30 12, 6, 3 units of Sonobe Module Origami
source: https://www.polypompholyx.com

The Sonobe unit takes the form of a parallelogram with 45 and 135-degree angles, divided by creases into two diagonal tabs at the ends and corresponding pockets within the inscribed center square. This system facilitates the construction of a myriad of three-dimensional geometric forms by interlocking tabs into the pockets of adjacent units. For instance, three interconnected Sonobe units fashion an open-bottomed triangular pyramid with an equilateral triangle serving as the open bottom and isosceles right triangles forming the other three faces. Notably, these pyramids can be arranged to point inwards, although assembly becomes more intricate. This unique feature, exemplified in "Toshie's Jewel," named after origami enthusiast Toshie Takahama, showcases a three-unit hexahedron constructed around the notional scaffold of a flat equilateral triangle. The resulting structure features two triangular pyramids joined at the base, forming a triangular bipyramid.

source: https://en.wikipedia.org/wiki/Sonobe

IV. Hierarchical Structure Modeling

Modular Origami can be described using a graph, where the edges represent the connections between individual origami units, and the vertices represent the points where these units meet. The initial phase in crafting modular origami involves the creation of unit origami. The illustration below provides an example of assembling an origami sonobe unit.

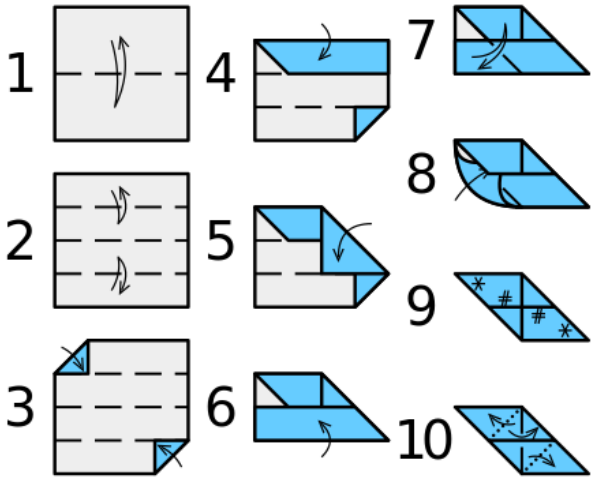

Fig 4.1. Assembling One Module Sonobe Origami
source: https://en.wikipedia.org/wiki/Sonobe

After we finish make some unit, the next step is to determine which type of Sonobe Modular Origami we will make. Each origami type is characterized by a specific arrangement of units, resulting in diverse geometric shapes. For instance, creating a model with three units forms a triangular bipyramid. Constructing a pyramid on each face of a regular tetrahedron, using six units, yields a cube or triakis tetrahedron (with modifications in faces, edges, and vertices due to the flattened central fold). Similarly, building pyramids on the faces of a regular octahedron or icosahedron, utilizing twelve and thirty units, respectively, results in a triakis octahedron or triakis icosahedron. The variety of available origami types showcases the versatility and creative possibilities within the realm of modular origami. Here's some example of Modular Origami Pattern that we can make

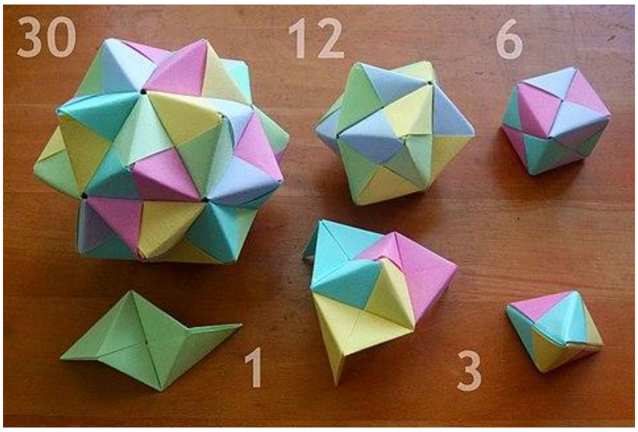

Fig 4.1. Sonobe Origami and Number of Module

1. Tetrahedron (Four Triangles)
   A tetrahedron is a polyhedron with four faces, each of which is an equilateral triangle.
   Properties: All edges and vertices are equivalent. It is the simplest of the platonic solids.
2. Cube (Six Squares)
   A cube, also known as a hexahedron, is a three-dimensional solid object bounded by six square faces, angles, and edges.
   Properties: All faces are identical squares. It has 12 edges and 8 vertices.
3. Octahedron (Eight Triangles)
   An octahedron is a polyhedron with eight faces, each of which is an equilateral triangle.
   Properties: It has 12 edges and 6 vertices. The dual of the cube, meaning the vertices of the octahedron correspond to the centers of the faces of the cube.
4. Dodecahedron (12 Pentagons):A dodecahedron is a polyhedron with 12 faces, each of which is a regular pentagon.
   Properties: It has 30 edges and 20 vertices. It is characterized by its regular pentagonal faces.
5. Icosahedron (20 Triangles)
   An icosahedron is a polyhedron with 20 equilateral triangle faces.
   Properties: It has 30 edges and 12 vertices. The dual of the dodecahedron, meaning the vertices of the icosahedron correspond to the centers of the faces of the dodecahedron.
6. Heptahedron (Seven Faces)
   A heptahedron is a polyhedron with seven faces. It can have various face shapes, and there is no unique heptahedron as in the case of the platonic solids.
   Properties: The number of edges and vertices depends on the specific configuration.
7. Icosidodecahedron (Twelve Equilateral Triangles and Twenty Equilateral Triangles)
   An icosidodecahedron is a polyhedron with twelve regular pentagonal faces and twenty equilateral triangle faces.
   Properties: It has 60 edges and 30 vertices.
8. Rhombic Triacontahedron (30 Rhombus Faces)
   A rhombic triacontahedron is a polyhedron with 30 rhombus faces. Each face is an equilateral rhombus.
   Properties: It has 60 edges and 32 vertices.
9. Pentagonal Hexecontahedron (60 Regular Pentagons)
   A pentagonal hexecontahedron is a polyhedron with 60 regular pentagonal faces.
   Properties: It has 120 edges and 62 vertices.
10. Snub Cube (38 Faces of Various Shapes)
    A snub cube is an Archimedean solid with 38 faces of various shapes, including squares and equilateral triangles.
    Properties: It has 60 edges and 24 vertices.
11. Great Stellated Dodecahedron (12 Pentagrammic Faces)

A great stellated dodecahedron is a Kepler-Poinsot solid with 12 pentagrammic faces.
Properties: It has 30 edges and 12 vertices.

12. Small Rhombihexahedron (12 Rhombus Faces):A small rhombihexahedron is a polyhedron with 12 rhombus faces.
Properties: It has 24 edges and 14 vertices.

Table 4.1 Correlation between The Number of Origami Needed to Create One Unit of Modular Origami and its Faces, Edges and Vertices

Number of Units	**Faces**	**Edges**	**Vertices**
3	6	9	5
6	12	18	8
12	24	36	14
30	60	90	32
90	180	270	92
120	240	360	122
270	540	810	272

From the table above we know that to make modular origami with s units individual origami paper, we will produce $2s$ faces, $3s$ edges, and $s + 2$ vertices.

B. Dodecahedron Of Cubes

A dodecahedron is a polyhedron with twelve flat faces, each of which is a regular pentagon. It is one of the five Platonic solids and exhibits unique symmetry properties.

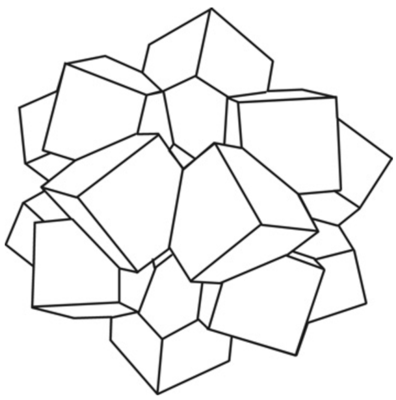

Fig 4.1. Dodecahedron Of Cubes

That Modular Origami has the vertices and edges of the graph correspond to the vertices and edges of the dodecahedron, respectively, so that if (u, v) is a directed edge of the graph, then a corner of the cube at u is inserted into an inverted corner of the cube at v. For an undirected graph $G = (V, E)$ and degrees $(d_v \mid v \in V)$, we may be given a collection of in-degree/out-degree pairs. The edges adjacent to vertex v are assigned a direction so that the in-degree/out-degree pair of v will be (i_s, o_s).

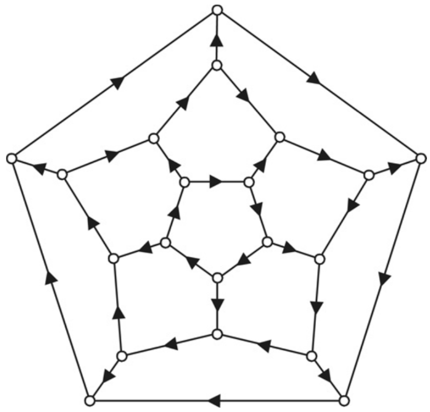

Fig 4.2. Dodecahedron Graph

Therefore, considering a given graph and a specific in-degree/out-degree sequence, a partial assembly can be defined as the mapping function f, along with the orientation of edges incident to U. A full assembly represents a complete configuration where U encompasses all vertices. An extension of a partial assembly, A1, compared to another partial assembly, A2, implies that every vertex belonging to A2 is also present in A1, and each edge in A2 maintains the same orientation in both assemblies.

The assembly process can be characterized as a sequential progression of partial assemblies, denoted as A0, A1, ..., An. Here, A0 is a trivial assembly with U being an empty set, An signifies a full assembly, and for each i = 1, 2, ..., n, the partial assembly Ai extends Ai−1 by incorporating a single vertex.

V. Conclusion

There are applications of graph theory in fields other than maths and science, with one of its utilizations shown in this paper. Modular Origami, when viewed through the lens of graph theory, unveils a rich and interconnected world where the assembly process can be systematically analyzed. By representing the connections between individual origami units as edges and the points where these units meet as vertices, we gain a powerful mathematical framework for understanding the quantitative aspects of modular origami assembly.

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Pernyataan

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Bandung, 11 Desember 2023

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