Derivation of Maekawa's Theorem Using Induction and Graph Theory

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***Abstract*—Origami as an artform has been rapidly developing since the application of mathematical tools in model designing in the 1980s. Methods commonly develop a model's crease patterns, whereby creases are represented as a graph with vertices and edges. One theorem that a flat-foldable model satisfies is Maekawa's Theorem. This paper derives Maekawa's Theorem using induction and graph theory.**

***Keywords*—Graph, Induction, Maekawa's Theorem, Origami Design**

I. Introduction

Origami is a form of art which involves folding paper, usually a square sheet, into a target model. These models range from traditional cranes to complex dragons, but a majority of origami techniques were developed relatively recently. In the 1970s, insects were generally thought to be very hard to design and fold, with at least one book claiming that an origami grasshopper was impossible to make using a single square. However, in the late 1970s and early 1980s, origami artists began using mathematical tools to approach model designing. Notable artists include Jun Maekawa and Fumiaki Kawahata. This led to an outburst of increasingly complex models and another method of model designing, discovered independently by two origami artists: Robert J. Lang and Toshiyuki Meguro. This method is known as circle-river packing, which takes a weighted tree representing the bodies and flaps of a folded model and generates a crease pattern to approximate the model. This approximation is also called a base, where a single base can continue to be folded into multiple distinct models.

The circle-river packing method of designing a base involves three major steps. First, a weighted tree is drawn to represent the model. Second, circles are packed onto a square sheet of paper to represent the vertices of the tree of degree one (representing a flap). Third, rivers and supporting creases are placed such that the crease is flat-foldable. Flat-foldable refers to the quality of a model being fully flat; that is, all faces of the paper lie on a single plane. One challenge in circle-river packing is ensuring a given crease pattern yields a flat-foldable model, and while this problem is NP-complete, the opposite can be checked [1]. For a given model to be flat-foldable, it must obey Maekawa's Theorem [2].

Deriving Maekawa's Theorem can be done using induction and graph coloring, by considering the graph representation of a crease pattern.

II. Theories

A. Induction

Induction is a method of proving statements about discrete objects and integers. Induction involves two steps, the basis step and the induction step. The basis is a true statement to which further statements can be reduced to. The induction state involves assuming that the statement is true for some discrete condition, e.g., $x = n$, and showing that the statement holds for $x = n+1$ in order to prove that the statement is true for any x. If the statement remains true at $x = n+1$, then the statement has been proven using induction.

B. Graph

A graph is a collection of vertices and edges to visually represent discrete objects and the relationships between them [3].

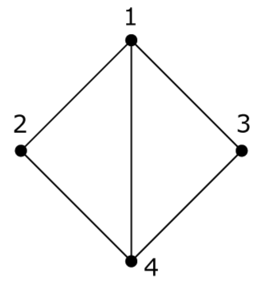

Fig. 1 Simple Graph

In Graph Theory, the following terms are often used:

1. Vertex
   A vertex is a discrete object represented by the graph.
2. Edge
   An edge represents a connection or relationship between two vertices.
3. Degree
   The degree of a vertex is an integer indicating the number of edges connected to the vertex.
4. Simple Graph
   A simple graph is a graph without parallel edges (more than one edge that connects any two vertices).

5. Multigraph
   A multigraph is a graph that contains parallel edges
6. Pseudograph
   A pseudograph is a graph that contains at least one edge connecting one vertex to itself (loop).
7. Path
   A path is a sequence of edges, where the target vertex of the first edge is the source vertex of the next edge.
8. Circuit
   A circuit is a path which ends at the starting vertex.
9. Dual Graph
   The dual of graph G is the graph formed by considering the faces of G as vertices and the edges of G between two faces as the edges connecting the vertices represented by those two faces.

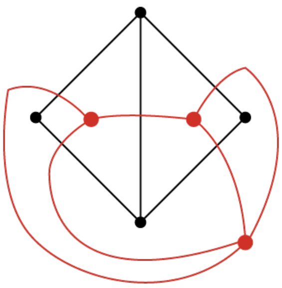

Fig. 2 Dual Graph of Fig. 1

10. Bipartite Graph
    A bipartite graph is a graph that can be divided into two sets, where vertices in the same set do not have edges that connect each other. Every cycle in a bipartite graph has an even length.

C. Euler Circuit

An Euler circuit is a circuit which visits every edge of a graph exactly once. If a graph has an Euler circuit, then either all of its vertices have an even degree or it only has two odd-degree vertices whereas the rest are even [3].

D. Graph Coloring

The nodes of a graph can be colored such that no two adjacent vertices have the same color. The minimum number of colors needed to color a graph is known as the chromatic number of a graph. A bipartite graph has a chromatic number of 2.

E. Origami

The following are common origami terms:

1. Mountain Fold
   A mountain fold is a fold which forms an angle greater than 180°, as measured from the face facing the observer. In other words, the moving face of a mountain fold moves away from the observer, forming a mountain. By convention, mountain folds are represented by straight unbroken lines on a paper.
2. Valley Fold
3. A mountain fold is a fold which forms an angle less than 180°, as measured from the face facing the observer. In other words, the moving face of a mountain fold moves towards the observer, forming a valley. By convention, mountain folds are represented by straight dashed lines on a paper.
4. Crease Pattern
   The crease pattern of a model is the set of resulting lines visualized upon unfolding a model and laying the paper flat in its original form (commonly a square sheet). The lines of a crease pattern may show mountain folds and valley folds.
5. Base
   The base of a model is an approximation of the model, commonly flat-folded and possessing the same number of notable features (such as amount of flaps, flap lengths, and flap widths).
6. Flat-Foldable
   A flat-foldable model is a model in which all faces of the paper lie on a single plane.

F. Maekawa's Theorem

At every vertex, the difference between the number of mountain folds and valley folds is 2 [2]. Formally, it is written as (1):

$$M - V = \pm 2$$

Equation (1), where M is the number of mountain folds and V is the number of valley folds.

III. Derivation of Circle-River Packing Rules

A. Representation of Crease Patterns

Origami crease patterns can be described using a graph, where the edges represent folds and the vertices represent where creases meet. For the purpose of circle-river packing, crease patterns must be flat-foldable to prevent the complication of convex and concave vertices inside the model. As such, one can attempt to represent the crease pattern of a simple three-flap base as Fig. 3.

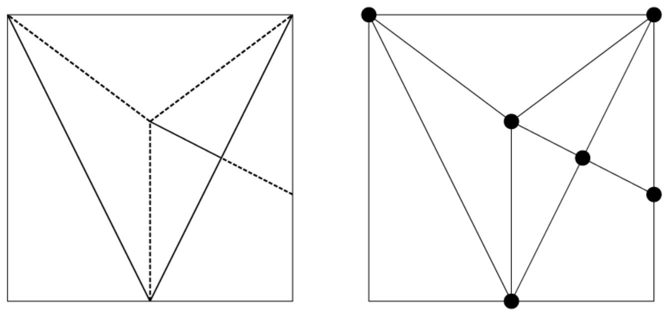

Fig. 3 Three-Flap Base and Associated Graph

An initial observation can be made: anywhere two edges meet on the plane is a vertex, and hence the graph is always drawn such that its edges intersect only on a common vertex. Therefore, all graphs made from flat-foldable crease patterns are planar graphs.

However, the graph in Fig. 3 does not suit the definition of a crease pattern's representation, because the vertices at the edge of the paper are not the result of two creases meeting (the paper's edge is unfolded). One must therefore revise the graph. One way might be to not consider the meeting of a fold and the paper's edge as a vertex, however this is problematic because it violates the definition of an edge and the definition of a crease pattern's representation, i.e., an edge connects two vertices, and a fold is an edge. An alternative way is to consider the paper's four edges as one continuous point, where all folds that meet the paper's edge meet. This is a sufficient rule, because it satisfies all definitions. The revised graph becomes Fig. 4.

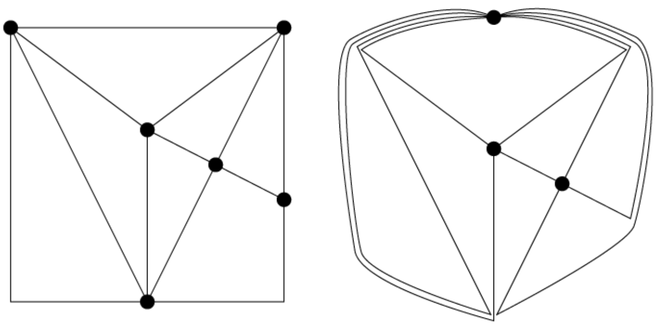

Fig. 4 Revised Graph of the Three-Flap Base

As a consequence, the graph of only a single fold must contain a loop, as the crease pattern consists of one fold connecting one edge of the paper to another, as shown in Fig. 5.

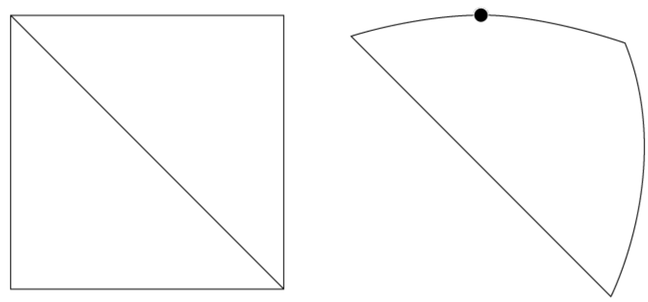

Fig. 5 Graph of Single Fold

B. Weak Form of Maekawa's Theorem

Take the case of a paper with one fold, e.g., a model made of a single diagonal fold.

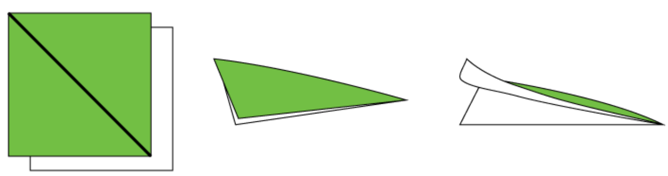

Fig. 6 Simple Fold

Its graph is depicted in Fig. 5, which consists of one vertex of degree 2. This shall be the basis for induction. That is, for a flat-foldable crease pattern representing one fold, there is an even number of edges connected to all its vertices. This is true regardless of where the fold is made. The hypothesis is that for any flat-foldable crease pattern, there is an even number of edges connected to all its vertices.

To begin the induction step, it is assumed that for a crease pattern representing two folds, there is an even number of edges connected to all its vertices. One possible crease pattern with two folds is shown in Fig. 7.

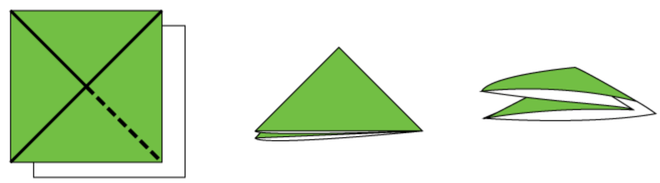

Fig. 7 Two Folds

This is easily verifiable by looking at the graph associated with such a crease pattern, as shown in Fig. 8.

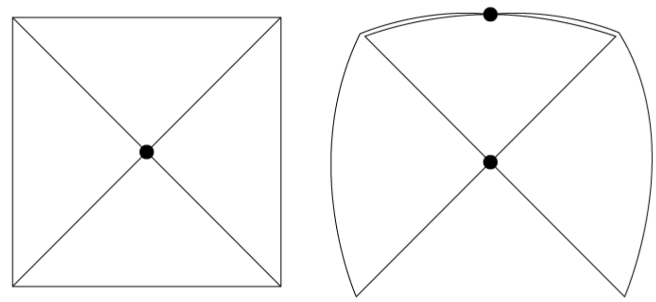

Fig. 8 Graph of Two Folds

If the hypothesis is true, then there must also be an even number of edges connected to all the vertices of a crease pattern with three folds. One can generate a three-fold crease pattern by folding an additional flap at the point where the two folds meet in Fig. 7, as depicted in Fig. 9.

Fig. 9 Adding One Fold to Two Folds

By unfolding the paper and constructing the graph, one can verify that every vertex is of an even degree. Therefore, the hypothesis is true.

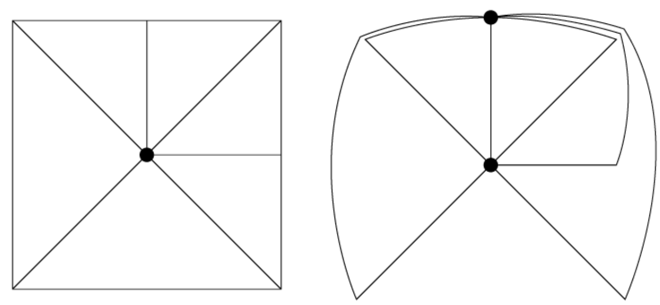

Fig. 10 Graph of Three Folds

C. Coloration of Flat-Folded Graphs

As a direct consequence of every vertex being an even degree, all flat-foldable crease patterns have Euler circuits. From here, it can be proven that it is possible to color the faces of a crease pattern using only two colors without any two adjacent faces having the same color.

Let G be the graph associated with a flat-foldable crease pattern. One can then construct the dual graph of G – G'. If the faces of G can be colored using only two colors, then the vertices of G' can be colored using only two colors; i.e., it has a chromatic number of 2. In other words, G' should be bipartite. One can prove this by showing that the opposite yields a contradiction [4].

If G' is not bipartite, then G' contains an odd cycle. If G' contains an odd cycle, then there exists a vertex in G with an odd degree (in order to connect an odd number of faces). However, as all vertices in G has an even degree (from III.B.), G' cannot contain an odd cycle. Therefore, G' cannot be not bipartite; G' is bipartite. Therefore, it is possible to color every face of a flat-foldable crease pattern using only two colors without neighboring faces having the same color.

D. Maekawa's Theorem

Taking the definition of a flat-foldable model, it is clear that a fold reverses the orientation of a face, as shown in Fig. 10.

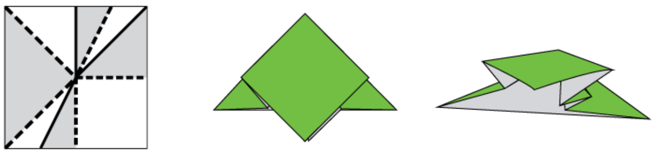

Fig. 11 Orientation of Faces around a Fold

If the angle between the top flap and the bottom flap were not zero, the model would not, by definition, be flat. As every fold is the locally the same (a region of paper that is folded), it follows that the two faces divided by every fold has reverse orientations. In other words, the orientation of every face is different from the orientation of its neighboring faces. Therefore, one can use the fact proven in III.C. to further state that the two colors represent a face's orientation.

It logically follows that every face about a vertex alternates between facing upwards and facing downwards. Because every face either faces upwards or downwards of the plane, every fold reverses the orientation of the paper, and there are an even number of folds around every vertex, one can recognize that save for the topmost folds, it is possible to pair every mountain and valley fold around a vertex.

As the topmost folds define the upper face around the vertex, they must be valley folds relative to the inner face of the fold. Relative to the outer face of the fold, however, they must be mountain folds. What this means is that around any vertex, there are two folds of the same type which do not have a pair. Consequently, the difference between the number of mountain folds and valley folds is two, which is what Maekawa's Theorem states.

IV. Conclusion

There are applications of graph theory and induction in fields other than maths and science, with one of its utilizations shown in this paper. Maekawa's Theorem, as proved using graph theory and induction in this paper, is further used in conjunction with other mathematical tools such as Kawasaki's Theorem and the Augmented Lagrangian method in order to develop algorithms for optimum circle-river packing in complex origami design [5].

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Pernyataan

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