Solving *the Five Card Trick* with Concepts of Discrete Mathematics

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***Abstract*—Mathematics has been used, both implicitly and explicitly, to create and dissect magic tricks and card games for centuries. In this paper, we will observe one example of a simple card trick created to teach and explore some fundamental concepts of discrete mathematics, including permutations, combinatorics, the pigeonhole principle, and number theory. It is the hope of the author that this paper will serve an interesting and entertaining analytical exercise for both the author and the reader and that this paper will leave the reader with a basic understanding of some concepts of discrete mathematics.**

***Keywords*—cards, combinatoric, number theory, permutation.**

I. Introduction

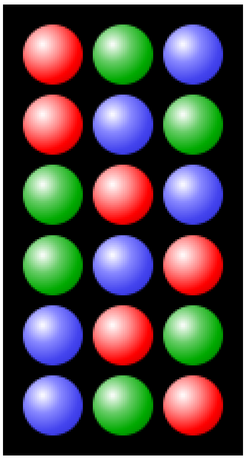
Before we begin to disassemble and explore the concepts of discrete mathematics that we will use to solve this card trick, we must first "set the scene" and explain what the trick is. A magician and his assistant perform in front of a large crowd. The assistant turns away from the audience to "gather his mystical energies" as the magician pulls out a deck of cards and asks a random member of the crowd to choose 5 cards at random. Once chosen, the audience member shows the rest of the audience the cards he/she has chosen. The magician then picks one card out of the hand and hands it to the audience member, who is instructed to show it to the audience and keep the card in his/her pocket. The magician then instructs the assistant to turn around and hands him the 4 remaining cards. After observing them with his "mind's eye", the assistant correctly deduces the chosen card.

Of course, this magic trick does not require the use "mystical energies channeled through a mind's eye". In fact, this trick was not created by a magician at all. Invented by mathematician and professor Fitch Cheney, the original trick appeared in 1950 in the book Math Miracles. Originally conceived as a magic trick conveyed through a telephone, the basics of the trick (the deck of cards, the five-card hand, the 'secret' card) remained the same throughout the years [1]. It was largely ignored until 1986, when Art Benjamin showed this trick at the Hampshire College Summer Studies in Mathematics. Several mathematicians have since analyzed it, discovered strategies, and even performed it in front of colleagues, students, and anyone else willing to spare the time [2].

As stated before, several strategies have been developed in performing this card trick. The strategy that we will be exploring utilizes permutations, combinations, and number theory. Before we outline and explain this strategy, we will outline the basics of some of these concepts.

II. THEORETICAL FRAMEWORK

A. Permutations



Example of permutation. Each row is a different permutation of the order of three distinct balls.

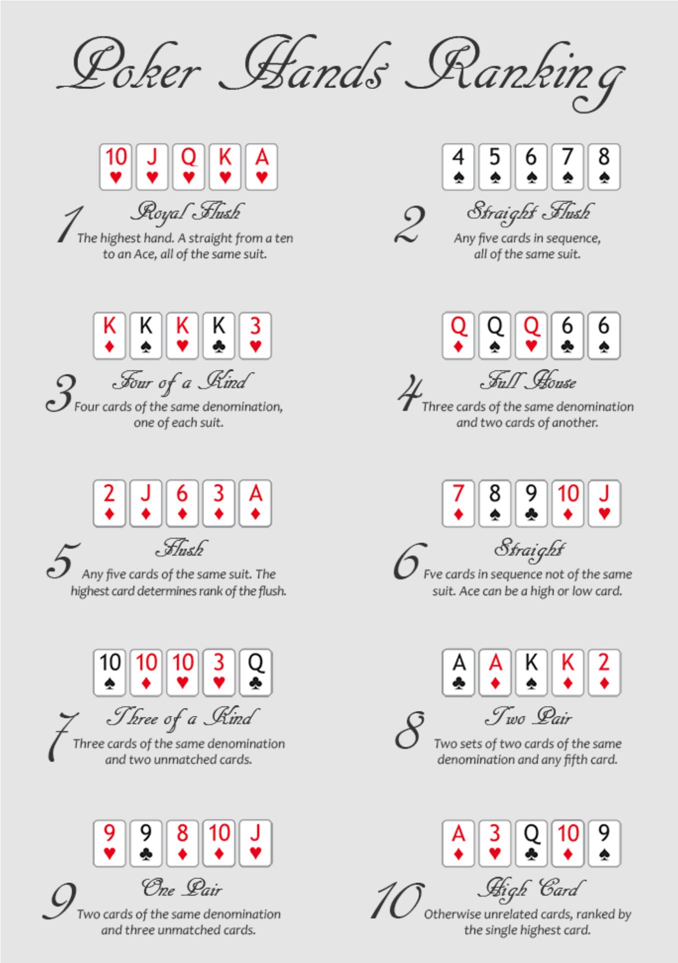
In mathematics, a permutation of a set is, loosely speaking, an arrangement of its members into a sequence or linear order, or if the set is already ordered, a rearrangement of its elements. The word "permutation" also refers to the act or process of changing the linear order of an ordered set [3]. The number of permutations of n distinct objects is n factorial, usually written as n!, which means the product of all positive integers less than or equal to n.

A weaker meaning of the term permutation, sometimes used in elementary combinatorics texts, designates those ordered arrangements in which no element occurs more than once, but without the requirement of using all the elements from a given set. These are not permutations, except in special cases, but are natural generalizations of the ordered arrangement concept. Indeed, this use often involves considering arrangements of a fixed length k of elements taken from a given set of size n, in other words, these k-permutations of n are the different ordered arrangements of a k-element subset of an n-set (sometimes called variations or arrangements in the older literature). These objects are also known as partial permutations or as sequences without repetition, terms that avoid confusion with the other, more common, meaning of "permutation". The number of such k-permutations of n is denoted variously by such symbols as ${}_n\boldsymbol{P}_k$, ${}^n\boldsymbol{P}_k$, $\boldsymbol{P}_{n,k}$, or $\boldsymbol{P}$ (n,k), and its value is given by the product

$$\boldsymbol{P}(n,k) = n * (n - 1) * (n - 2) * \ldots * (n - k + 1)$$

Or more commonly known as n! / (n - k)!.

B. Combinatorials



Example of combinations. Note how the position of the cards do not differentiate between different hands.

In mathematics, a combination is a selection of items from a collection, such that the order of selection does not matter (unlike permutations). For example, given three fruits, say an apple, an orange and a pear, there are three combinations of two that can be drawn from this set: an apple and a pear; an apple and an orange; or a pear and an orange. More formally, a k-combination of a set S is a subset of k distinct elements of S. If the set has n elements, the number of k-combinations is represented by the notations ${}_n\boldsymbol{C}_k$, ${}^n\boldsymbol{C}_k$, $\boldsymbol{C}_{n,k}$, or $\boldsymbol{C}$ (n,k) and is equal to

$$\boldsymbol{C}(n,k) = \boldsymbol{P}(n,k) / k!$$

Or more commonly know as n! / k! * (n - k)!

Although the set of three fruits was small enough to write a complete list of combinations, this becomes impractical as the size of the set increases. For example, a poker hand can be described as a 5-combination (k = 5) of cards from a 52 card deck (n = 52). The 5 cards of the hand are all distinct, and the order of cards in the hand does not matter.

C. Number Theory

Number theory (or arithmetic or higher arithmetic in older usage) is a branch of pure mathematics devoted primarily to the study of the integers and integer-valued functions. Number theorists study prime numbers as well as the properties of mathematical objects made out of integers (for example, rational numbers) or defined as generalizations of the integers (for example, algebraic integers).

One of the fundamental areas of study for number theory involve modular arithmetic. In mathematics, modular arithmetic is a system of arithmetic for integers, where numbers "wrap around" when reaching a certain value, called the modulus. The modern approach to modular arithmetic was developed by Carl Friedrich Gauss in his book Disquisitiones Arithmeticae, published in 1801. A familiar use of modular arithmetic is in the 12-hour clock, in which the day is divided into two 12-hour periods. If the time is 7:00 now, then 8 hours later it will be 3:00. Simple addition would result in 7 + 8 = 15, but clocks "wrap around" every 12 hours. Because the hour number starts over after it reaches 12, this is arithmetic modulo 12.

Given an integer n > 1, called a modulus, two integers are said to be congruent modulo n, if n is a divisor of their difference (i.e., if there is an integer k such that a − b = kn).

Congruence modulo n is a congruence relation, meaning that it is an equivalence relation that is compatible with the operations of addition, subtraction, and multiplication. Congruence modulo n is denoted:

$$a \equiv b \pmod{n}$$

The parentheses mean that (mod n) applies to the entire equation, not just to the right-hand side (here b). This notation is not to be confused with the notation b mod n (without parentheses), which refers to the modulo operation. Indeed, b mod n denotes the unique integer a such that $0 \le a < n$ and $a \equiv b \pmod{n}$ (i.e., the remainder of b when divided by n).

The congruence relation may be rewritten as

$$a = kn + b$$

explicitly showing its relationship with Euclidean division. However, the b here need not be the remainder of the division of a by n. Instead, what the statement $a \equiv b \pmod{n}$ asserts is that a and b have the same remainder when divided by n.

III. The Five Card Trick Strategy

IV. Conclusion

V. Acknowledgment

References

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Pernyataan

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

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