Combinatorial Game Theory to Analyze Chess Endgames

Daffa Ananda Pratama Resyaly – 13519107
Program Studi Teknik Informatika
Sekolah Teknik Elektro dan Informatika
Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia
13519107@std.stei.itb.ac.id

***Abstract*—Chess is a board game of strategic skill for two players, played on chessboard, a checkered board with 64 squares arranged in 8x8 grid. Each player begins the game with sixteen pieces that are moved and used to capture opposing pieces according to precise rules. The object is to put the opponent's king under a direct attack from which escape is impossible (*checkmate*). In this paper, I attempt to analyze Chess endgames using combinatorial game theory.**

***Keywords*—Chess, Combinatorial, Endgame, Theory**

I. Introduction

Chess is a two-player strategy board game played on a chessboard, a checkered board with 64 squares arranged in an 8x8 grid. Chess first appeared in India about the 6th century ad and by the 10th century had spread from Asia to the Middle East and Europe. Since at least the 15th century, chess has been known as the "royal game" because of its popularity among the nobility. Rules and set design slowly evolved until both reached today's standard in the early 19th century. Once an intellectual diversion favored by the upper classes, chess went through an explosive growth in interest during the 20th century as professional and state-sponsored players competed for an officially recognized world championship title and increasingly lucrative tournament prizes. Organized chess tournaments, postal correspondence games, and Internet chess now attract men, women, and children around the world.

By convention, this game pieces are divided into two sets, white and black. Each set consists of sixteen pieces, which are eight pawns, two knights, two bishops, two rooks, one queen, and one king.

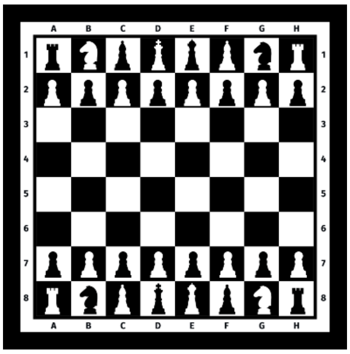

Figure 1. Chessboard and Initial Position of Chess Game (Source : https://images.app.goo.gl/CSE7FRP8gBsuzDbr8)

Chess is played on a board of 64 squares arranged in eight vertical rows called files and eight horizontal rows called ranks. These squares alternate between two colors: one light, such as white, beige, or yellow; and the other dark, such as black or green. The board is set between the two opponents so that each player has a light-colored square at the right-hand corner.

White moves first, after which players alternate turns, moving one piece per turn (except for castling, when two pieces are moved). A piece is moved to either an unoccupied square or one occupied by an opponent's piece, which is captured and removed from play. With the sole exception of *en passant*, all pieces capture by moving to the square that the opponent's piece occupies.

Moving is compulsory. It is illegal to skip a turn, even when having to move *is detrimental*. A player may not make any move that would put or leave the player's own king in check. If the player to move has no legal move, the game is over. The result is either checkmate (a loss for the player with no legal move) if the king is in check, or stalemate (a draw) if the king is not in check.

II. Fundamental Theory

A. Combination

1) Definition

Combinatorial is an area of mathematics primarily concerned with the arrangement of, operation on, and selection of discrete mathematical elements belonging to finite sets or making up geometric configurations.

Combinatorial, insofar as an area can be described by the types of problems it addresses, is involved with

- The *enumeration* (counting) of specified structures, sometimes referred to as arrangements or configurations in a very general sense, associated with finite systems
- The *existence* of such structures that satisfy certain given criteria
- The *construction* of these structures, perhaps in many ways
- *Optimization*, finding the "best" structure or solution among several possibilities, be it the "largest", "smallest" or satisfying some other *optimality criterion*.

Combinatorial is well known for the breadth of the problems it takes. Combinatorial problems arise in many areas of pure mathematics, notably in algebra, probability theory, topology, and geometry, as well as in its many application areas.

2) Inclusion and Exclusion Principle

The principle of inclusion and exclusion (PIE) is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

An underlying idea behind PIE is that summing the number of elements that satisfy at least one of two categories and subtracting the overlap prevents double counting. For instance, the number of people that have at least one cat or at least one dog can be found by taking the number of people who own a cat, adding the number of people that have a dog, then subtracting the number of people who have both.

PIE is particularly useful in combinatorics and probability problem solving when it is necessary to devise a counting method that ensures an object is not counted twice.

In the case of objects being separated into two (possibly disjoint) sets, the principle of inclusion and exclusion states

$$|A\cup B|=|A|+|B|-|A\cap B|,$$

where $|S|$ denotes the cardinality, or number of elements, of set S in set notation.

As a Venn diagram, PIE for two sets can be depicted easily:

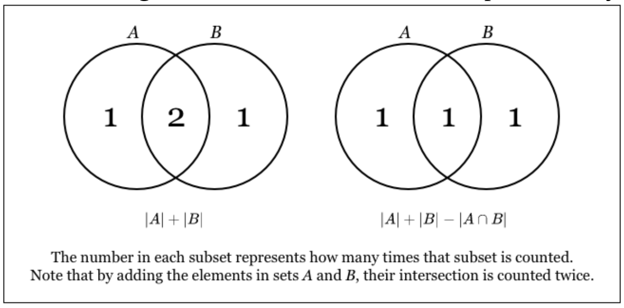

Figure 2. Venn Diagram of PIE for two sets
(Source : https://brilliant.org/wiki/principle-of-inclusion-and-exclusion-pie/)

If there are three sets, the principle of inclusion and exclusion states

$$|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|.$$

We can verify these statements for ourselves by considering the Venn diagram of events:

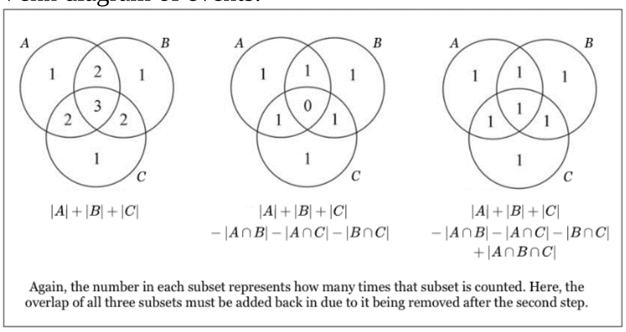

Figure 3. Venn Diagram of PIE for three sets
(Source : https://brilliant.org/wiki/principle-of-inclusion-and-exclusion-pie/)

3) Permutations

A permutation, also called an "arrangement number" or "order," is a rearrangement of the elements of an ordered list S into a one-to-one correspondence with S itself. The number of permutations on a set of n elements is given by n!. For example, there are 2!=2·1=2 permutations of {1,2}, namely {1,2} and {2,1}, and 3!=3·2·1=6 permutations of {1,2,3}, namely {1,2,3}, {1,3,2}, {2,1,3}, {2,3,1}, {3,1,2}, and {3,2,1}. The number of ways of obtaining an ordered subset of k elements from a set of n elements is given by

$$_nP_k \equiv \frac{n!}{(n-k)!}$$

Figure 4. Permutation Basic Formula
(Source : https://mathworld.wolfram.com/images/equations/Permutation/NumberedEquation1.gif)

where n! is a factorial. For example, there are 4!/2!=12 2-subsets of {1,2,3,4}, namely {1,2}, {1,3}, {1,4}, {2,1}, {2,3}, {2,4}, {3,1}, {3,2}, {3,4}, {4,1}, {4,2}, and {4,3}. The unordered subsets containing k elements are known as the k-subsets of a given set.

4) Combination

Combination is the number of ways of picking k unordered outcomes from n possibilities. Also known as the binomial coefficient or choice number and read "n choose k,"

$$_nC_k \equiv \binom{n}{k} \equiv \frac{n!}{k!\,(n-k)!},$$

Figure 5. Combination Basic Formula
(Source : https://mathworld.wolfram.com/images/equations/Combination/NumberedEquation1.gif)

where n! is a factorial. For example, there are (4; 2) = 6 combinations of two elements out of the set {1,2,3,4}, namely {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, and {3,4}. These combinations are known as k-subsets.

B. Combinatorial Game Theory

Combinatorial game is two-person game with perfect information and no chance moves (no randomization like coin toss is involved that can effect the game). These games have a win-or-lose or tie outcome and determined by a set of positions, including an initial position, and the player whose turn it is to move. Play moves from one position to another, with the players usually alternating moves, until a terminal position is reached. A terminal position is one from which no moves are possible. Then one of the players is declared the winner and the other the loser. Or there is a tie (Depending on the rules of the combinatorial game, the game could end up in a tie. The only thing that can be stated about the combinatorial game is that the game should end at some point and should not be stuck in a loop. In order to prevent such looping situations in games like chess(consider the case of both the players just moving their queens to-and-fro from one place to the other), there is actually a "50-move rule" according to which the game is considered to be drawn if the last 50 moves by each player have been completed without the movement of any pawn and without any capture.

Consider a game, given a number of piles in which each pile contains some numbers of stones/coins. In each turn, player choose one pile and remove any number of stones (at least one) from that pile. The player who cannot move is considered to lose the game (ie., one who take the last stone is the winner).

As it can be clearly seen from the rules of the above game that the moves are same for both the players. There is no restriction on one player over the other. Such a game is considered to be impartial.

III. Methodology

A. Zero Game

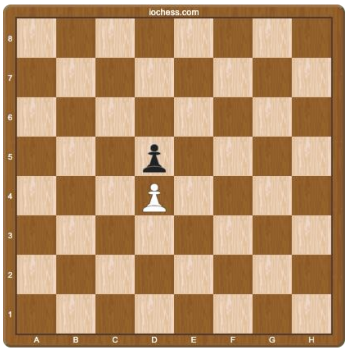

Figure 6. The Zero Game Condition

(Source : https://chessboardimage.com/8/8/8/3p4/3P4/8/8/8)

In Combinatorial Game Theory, the zero game, or 0, is defined as the game where neither player can make any move. Under normal play, the zero game is a second player win, because the first player cannot make any moves. In standard notation, 0 = {|}. The game above, which features two deadlocked pawns at mid board, is equal to the zero game under normal play, because neither black nor white have any moves.

Consider this condition now:

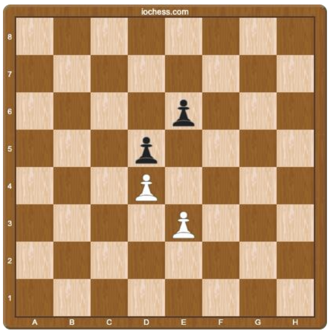

Figure 7. Condition 1

(Source : https://chessboardimage.com/8/8/4p3/3p4/3P4/4P3/8/8)

If white is to move, the result is a lost pawn, and black creates a passed pawn which will promote. If black is to move, the outcome is the opposite. At this point, we feel it is sensible to define creating a passed pawn that will promote as winning the game. With this definition in mind, we see that the position 1.2 is a second player win, and therefore must be equal to the zero game. With those positions in mind, it is worth asking whether other chess positions are equal to the zero game. The answer is yes, and many of the zero positions we found exhibit symmetrical characteristics. Take for example, the following pawn position, which is a second player win under normal play:

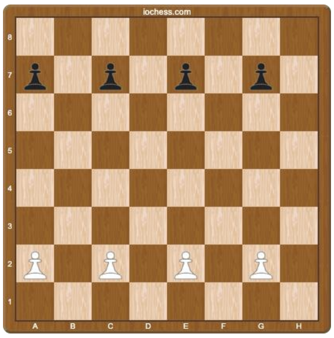

Figure 8. Condition 2

(Source : https://chessboardimage.com/8/p1p1p1p1/8/8/8/8/P1P1P1P1/8)

It is important to recognize positions that are equal to the zero game because these are second player wins–hence, you should avoid playing in them.

B. Some Nonzero Numbers : ±1 and ±2

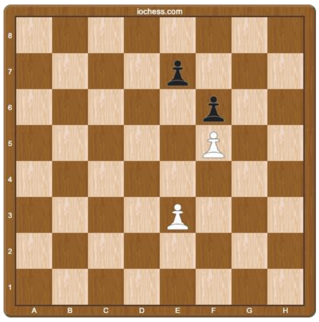

Figure 9. Game 1

(Source : https://chessboardimage.com/8/4p3/5p2/5P2/8/4P3/8/8)

In the game 1, Left can move to 0 and Right has no moves, so 1 is denoted {0|}. Hence, to construct the game 1 we first looked at a 0 position and modified it as can be seen in position 3.1. White can move to the zero game, but black's only moves lead to a lost e-pawn and a new white queen. In order for this game to completely make sense as 1, we will make another definition: we'll say that having only losing moves is the same as having no moves. This allows us to represent the above position as {0|} and call it 1. Given that the above position is equal to 1, we can now easily create a position equal to 2by moving white's e-pawn back to the second rank:

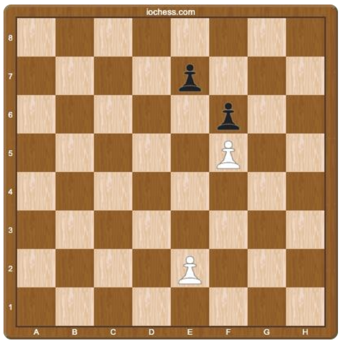

Figure 10. Game 2

(Source : https://chessboardimage.com/8/4p3/5p2/5P2/8/8/4P3/8)

2 is denoted {0,1‖}, and you can see that in the above position white can move to both 0 and 1, while black has only losing moves. The reader should note that it is now easy to construct−1 and−2 by switching the white and black pieces and rotating the board 180 degrees.

C. Some Infinitesimals:, ↑, ↓*

* is the game in which both left and right can move to the zero game. It is represented formally by {0‖0}. We found the following position to be equal to *:

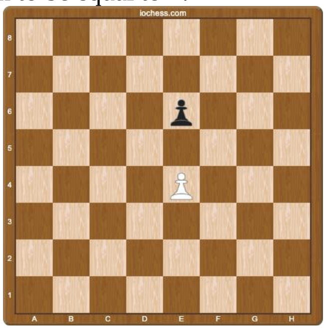

Figure 11. Game *
(Source : https://chessboardimage.com/8/8/4p3/8/4P3/8/8/8)

The game ↑ is denoted {0‖*}, and is a positive infinitesimal. Here is a position equal to ↑:

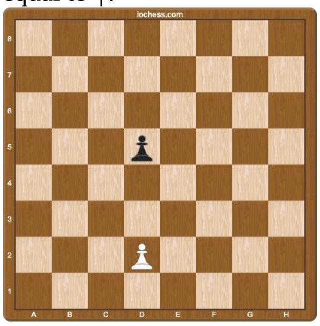

Figure 12. Game ↑
(Source : https://chessboardimage.com/8/8/8/3p4/8/8/3P4/8)

If we denote this game in the standard way, we get {0, *‖*}, but * is a reversible option for white. This is because the original game is greater than zero, and if white plays to * then black will win by moving to 0. Hence, the game's canonical form is {0‖*}, which equals↑. ↓can be constructed in the same way we described for obtaining−1 from 1.

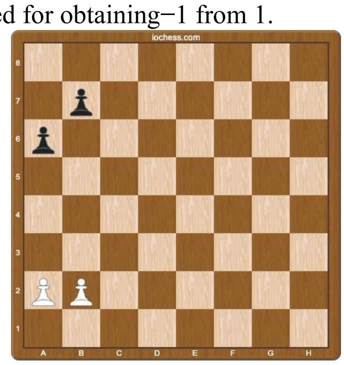

Figure 13. Game that also equals ↑
(Source : https://chessboardimage.com/8/1p6/p7/8/8/8/PP6/8)

Position 4.3 above is also equal to↑. An evaluation of the game tree will soon give us options that can be pruned by canonicalization. For white, a3 results in a symmetric position, which reduces to zero, because white can mirror any black reply with a pawn move of its own, eventually reducing to a mutual zugzwang. White can also force a zero position with b4, as black is forced to respond with b6, leading to white playing a4 and into a symmetrical game. Whites other move, a4, results in the hot game of {2‖0} with black to play, which is a reversible option, so it can be pruned from the game tree. 1. b3a5 2. a3 a4, loses for white, making b3 also a prunable option. Black's options are a little more limited. Playing b5 results in white being able to move to the 1 game, which black certainly does not want, and b6 results in a position that is greater than zero. The only move black really has is a5, which moves the game into a star position. This is because if black could move again to a4, then black moves to the zero game, whereas whites response to a4 would also result in the zero game. Thus, it is shown that this position can indeed be canonicalized to {0‖*}, or ↑.

D. Tiny 1

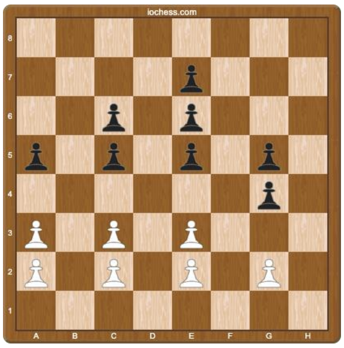

Figure 14. Tiny 1
(Source : https://chessboardimage.com/8/4p3/2p1p3/p1p1p1p1/6p1/P1P1P3/P1P1P1P1/8)

To construct Tiny 1, we look at games like the four pawn formations in position 5.1. The reader can check that, from left to right, the pawn formations are equal to{1‖0},{1‖−1},{1‖−2}, and{0‖−1}. Tiny 1 is denoted {0‖{0|−1}}, so in Tiny 1 white will be only be able to move to the zero game, while black's option will be to move to the rightmost pawn position above. Here is the position:

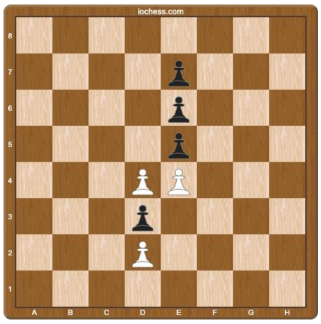

Figure 15. Game Tiny 1
(Source : https://chessboardimage.com/8/4p3/4p3/4p3/3PP3/3p4/3P4/8)

White's only reasonable move is to take black's pawn on the fifth rank, leading to a dead-locked zero position (note that white's option to advance the d-pawn is dominated because it leads to a passed pawn for black). Black can only take white's pawn on d4, which leaves the position $\{0 \| -1\}$, as desired. Tiny 1 is a positive infinitesimal, which means it should be a white win, which this position is. Tiny 1 is also a particularly small infinitesimal–it is infinitesimal with respect to ↑. Later, when we discuss game sums, we will show that this position is indeed smaller than ↑.

E. The Trebuchet Position, and Game Sums

Now we will bring kings onto the board. The trebuchet is an example of a reciprocal zugzwang position–a position in which both black and white would prefer not to move. In Position 6.1, the side to move loses the game, so we can say it is equal to the zero game.

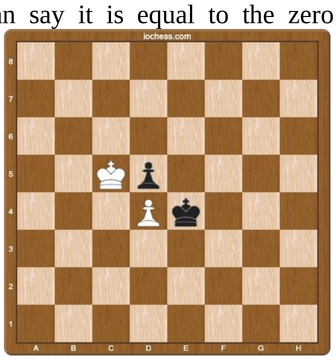

Figure 16. Trebuchet
(Source : https://chessboardimage.com/8/8/8/2Kp4/3Pk3/8/8/8)

It is easily checked that either side to move loses its pawn, and with correct play the other side's pawn will advance and promote, winning the game. Since this position equals the zero game, we can now combine it with the other positions we have previously constructed in order to form positions of chess that are combinatorial games under normal play. For example, let's look at what happens when we combine the trebuchet, Tiny 1 and ↓ :

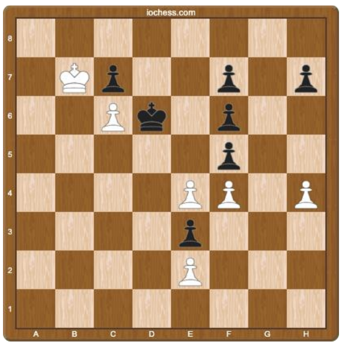

Figure 17. Trebuchet, Tiny 1, and ↓
(Source : https://chessboardimage.com/8/1Kp2p1p/2Pk1p2/5p2/4PP1P/4p3/4P3/8)

Tiny 1 is a positive infinitesimal, ↓is a negative infinitesimal, and the trebuchet is equal to zero. However, since Tiny 1 is infinitesimal with respect to both ↑ and ↓ , we would expect this position to be less than 0, or a win for black. It turns out that this is the case. If white is to move, white can move either in Tiny 1 or down. If white moves in Tiny 1, only the game ↓ remains. Black then moves ↓ to 0, and white is forced to play in the trebuchet and loses. If, on the other hand, white first moves ↓ to *, then black's winning response is to take white's pawn in Tiny 1, leaving the games $\{0 \| -1\}$ and *. There are two remaining moves before someone has to move in the trebuchet, and white must move, so white loses. If black moves first, the winning move is to take white's pawn in Tiny 1. This shows that our position is indeed less than 0. Of course, knowing our theory makes computing the outcome class of the position much quicker, because of the fact:

$$t1 + \downarrow < 0.$$

F. Strategies in Game Sums

Playing in game sums in chess endgames, we will advise two principles.

Principle 1. The number avoidance theorem says that in a game which is a sum of numbers and games that are not numbers, if a player can win, s/he can do it by not moving in the numbers. Hence, in chess endgames, when faced with a sum of games that are numbers and games that are not, do not move in the numbers.

Principle 2. When given the opportunity, one should move in the game with maximal temperature. The temperature of infinitesimals is 0, and the temperature of a switch game $\{x \| y\}$, $x > y$, is$(x-y)2$. This strategy is known as Hotstrat in Combinatorial Game Theory. Let's see the demonstration:

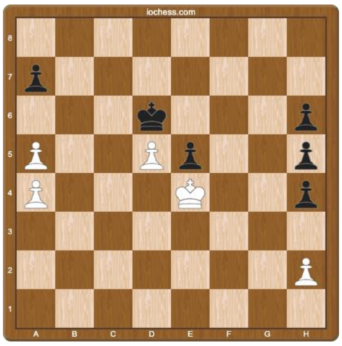

Figure 18. Demonstration
(Source : https://chessboardimage.com/8/p7/3k3p/P2Pp2p/P3K2p/8/7P/8)

In figure 18, we have two pawn switch games and the kings are involved in a trebuchet, which we'll think of as the zero game. Principle 1 says that neither player should move in 0. Principle 2 says to play in the hottest game, so let's calculate the temperature of the two switch games. The pawn position on the left is equal to $\{1 \| 0\}$, so its temperature is

$$(1-0)/2=1/2.$$

.The pawn position on the right is $\{0 \| -2\}$, which has temperature

$$(0 + 2)/2= 1$$

Principle 2 says to move in the game on the right. We can see that black will win this game no matter what, but white can lose by less pawn moves by moving to h2 instead of a6.

V. CONCLUSION

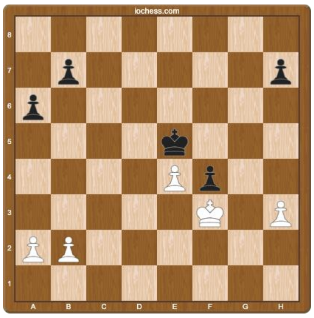

Figure 19. First to Move Wins
(Source :
https://chessboardimage.com/8/1p5p/p7/4k3/4Pp2/5K1P/PP6/8
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Here is an application to a real game in which both sides seem evenly matched. The black and white kings are locked in a trebuchet position and thus do not want to have to move, as moving would lead to the concession of a pawn–critical in an endgame like this one. Hence, we'll treat the trebuchet as a zero game and look at the sum of the two pawn games. The pawn game on the queenside is just figure 13, which is equal to $\uparrow$, whereas the game on the kingside canonicalizes to$\Downarrow ?(\{\downarrow \| 0, *\})$. Adding the respective values of the game sums gives us $\downarrow *$, which is confused with zero. This means that whoever is first to play should be able to make a winning pawn move. If white is first to play, white plays 1. h4, which moves to $\downarrow$ in the local pawn game, but moves to zero in the entire game, because the other position is equal to $\uparrow$. The game sum is now a second player win with Black to play. Black playing 1. ...a5 first achieves a similar effect, resulting in a black win. In terms of chess intuition, one can explain the merits of a5 and h4 as breaking up the opponents move choice, as pawns can choose to either move up one or two spaces on the first move. This fact is invaluable for wasting turns, so white wants to play 1. h4 and then 2. h5 immediately following, just as black wants to play 1. ...a5 and then 2. ...a4 next turn.

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PERNYATAAN

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

Jakarta, 11 Desember 2020

Ttd

Daffa Ananda Pratama Resyaly – 13519107