Not for Your Eyes Only: Hidden Fractal Patterns in Music

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***Abstract*—An art piece can be appreciated not just in a form of what you can see, but also in a form of what you can hear. The many definitions of the fractal suggest that 'the fractal is a self-similar object or figure that is simple, and yet infinitely complex'. This paper is to present an overview of fractals in mathematics and to show the fractal nature of music in some canonic works of Western music. The fact that fractals could be in realm of music is fascinating.**

***Keywords*—1/f noise, fractal, fractal music, fractal geometry**

I. Introduction

Fractals is one of the most amazing things we have found in the history of mankind. Previously unknown to human, its origin could already be seen in nature.

Fractals are commonly identifiable through the medium of sight. Trees, clouds, and almost all-natural objects are usually in a form of a fractal. The most famous example is the Mandelbrot Set. Music, however, is not identifiable through the medium of sight alone. Hidden beneath the musical work of past Western musician, another kind of fractal emerges. Although it cannot be seen by the naked eye, human can hear them. Through analysis, human now can identify them, using analogues and computer.

II. Fractal

Benoit Mandelbrot first wrote and coined in his book the term of fractal. It is written that the term 'fractal' is picked from a Latin word '*fractus*', meaning 'broken or fractured'. [2] The notion he created became an ideal way for a mathematical description for objects those have a recursion in their geometry. What Mandelbrot proposed was identifying irregular geometric objects and visualizing it using an algorithm on a computer. The computer-generated image was created, a set which contains an infinite number of other small sets, very similar to one another only differing in size. The image is called the Mandelbrot Set.

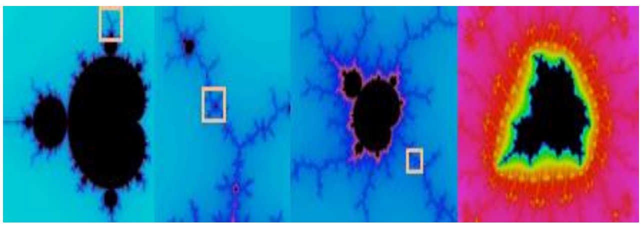

Figure 1. Quasi-self-similarity in the Mandelbrot set. [3]

Among the many definitions of the word 'fractal' created by mathematicians, all have a common ground that 'the fractal is a self-similar figure'. Fractals are usually found commonly in nature. The occurrence of fractal structure in nature has led several to state that fractal geometry is the geometry of nature. Mandelbrot himself stated that 'many facets of nature can only be described with the help of fractals.' He also stated that 'Nature's pattern is irregular and fragmented.' Thus, fractal objects include snowflakes, mountain ranges, rivers, and some plants. Fern is one of the examples.

Figure 2. A macro photography of a snowflake [4]

Figure 3. A fern [5]

In fractal, there are three type of self-similarity:

1. Quasi-self-similarity, in which the fractal appears approximately identical at different scales;
2. Exact self-similarity, a faithful copy of the objects as scale models;
3. Statistical self-similarity, in which the fractal has

statistical measures which are preserved across scales, as in nature. [6]

In mathematics, there exist a set which observed before the notation of 'fractals' even existed. The Cantor set is that set. It is formed from a line divided into three parts. The middle part of the thirds is then removed. The remaining part are replicated below the original line and the process of removing the middle thirds and replicating continue so on infinite times. The parts go infinitesimally smaller each repetition. This creates a 'saw-jagged-tooth' pattern.

Figure 4. The Cantor set [7]

Another famous fractal is the Waclaw Sierpinski triangle, also known as the Sierpinski gasket. This fractal is produced from an initial equilateral triangle. Then, the triangle is divided into four equal triangles. The middle one, which has its points in the center of the sides of the higher-order triangle is then removed. These operations are then repeated on the remaining triangles, creating an infinite number of triangles after infinite number of repetitions.

Figure 5. The Sierpinski triangle with 5 iterations. [8]

The fractals above belong to the group of iterated function systems (IFS). These IFS fractals are created iteratively through self-replication. There are two other types of fractals. Those are escape-time fractals (*e.g.* the Mandelbrot set) and random stochastically generated fractals (*e.g.* natural objects). These fractals are what we find in music.

III. Music and Its Terms

Before we dive deeper, we need to know the terms that will be used in this paper.

A. Canon

In music, a canon is a counterpoint-based compositional technique that employs a melody with one or more imitations of the melody played after a given duration. The initial melody is called the *leader* or *dux*. The following melody, which is played in a different voice, is called the *follower* or *comes*. The follower imitates the leader, either as an exact replication of its rhythms and intervals or some transformation of it. A popular example is the song "*Frère Jacques*", a nursery rhyme of French origin.

B. Bourrée

In music, a bourrée is an optional movement in the classical suite of dances. J.S. Bach, Handel, and Chopin wrote bourrées, although not necessarily made to be danced.

C. Sheet Music

In music, a sheet music is a handwritten or printed form of musical notation that uses musical symbols to indicate the melodies, rhythms or chords of a musical piece.

Figure 6. An example of a music sheet [9]

D. Musical Notation

In most classical music, the melody and the accompaniment parts are notated on the line of staff (a set of five horizontal lines and four spaces that each represent a different musical pitch or in the case of a percussion staff, different percussion instruments). The staff typically contains:

1. A clef, such as the bass clef 𝄢 or the treble clef 𝄞
2. A key signature, indicating the key.
3. A time signature, which typically has two number aligned vertically with the bottom indicating the note value that represents one beat and the top number indicating how many beats are in a bar. As an example, a time signature of $^4{}_4$ indicates that there are four quarter notes per bar.
4. Notes, a symbol denoting a musical sound that represent the pitch and duration of a sound in musical notation.

Figure 7. A simple grand staff

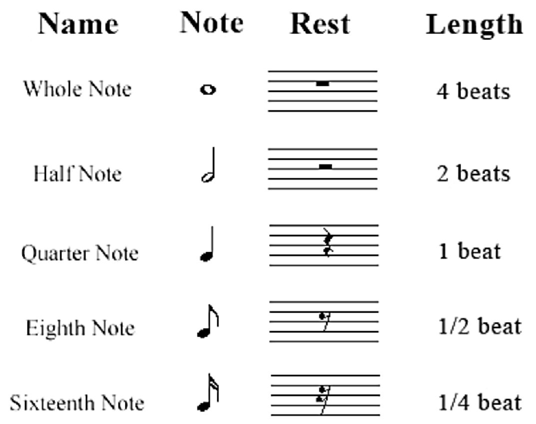

Figure 8. Notes, rests, and its length

IV. Fractals in Music

A. Fractals Analogues in Music

The properties such as self-similarity and scaling appear in many canonic works of Western music. As Mandelbrot stated, since music is part of human nature, he suggested that music displays fractal characteristics because music is hierarchical in nature. The best examples of fractals in music are those in canons and fugues. In a canon, fractals can be seen as be an exact replica or of its transformation. This is also connected with the existence of canons in different types.

For an example, we analyze the first bourrée of *Cello Suite No. 3* by Johann Sebastian Bach.

Figure 9. A portrait of Johann Sebastian Bach [10]

Harlan J. Brothers suggests that the first bourrée of *Cello Suite No. 3* is an example of structural scaling in respect to phrasing. Those phrasing may be visualized analogous to the Cantor set. In the first sixteen bars of the composition, it contains sequences of notes which relate to each other in a specific way. It has an A-A-B, where section B is twice as long as section A. Let *M1* be the basic model, which have two joint eighth notes (quavers) and a quarter note (crochet). *M1* is first repeated (*M2*), and then expanded into a phrase twice as long (*M3*). *M1, M2,* and *M3* form a set called *S1*. *S1* is repeated on the first bourrée (*S2*) and then also expanded and transformed to form *S3*. Therefore, the pattern is 2 bars – 2 bars – 4 bars. [11]

Figure 10. Analysis of the first 16 bars of the first bourrée of *Cello Suite No. 3* [11]

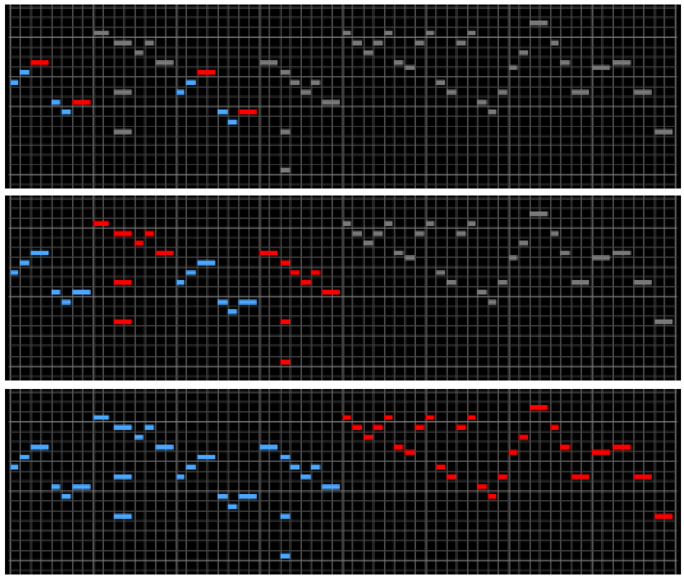

Figure 11. Graphical notations of the structural scaling seen in *Cello Suite No. 3* [11]

Another example can be taken from *Ecossaise WoO 83 No.1* by Ludwig van Beethoven.

Figure 12. A portrait of Ludwig van Beethoven [12]

The binary structure of the first cycle of the six *Ecossaises (WoO 83)* and the self-similarity of the motifs employed there. By analyzing thirty-two bars of the score, Larry Solomon distinguishes two-sections of equal length, marked *A* and *B*. Each of this section represents two eight-bar periods. These eight-bar is then divided again into two four-bar period, which is again, a binary. The four-bar are divided into two two-bar, which then divided into one-bar motifs marked as *M* as also their transformation. These section divisions have a meaning. In other words, each division of the thirty-two bar constitutes a smaller replica, smaller bars, of a larger unit. This also means that the thirty-two bar is also analogous to the Cantor set. [13]

Figure 13. Analysis of the 32 bars of *Eccossaise WoO 83 No. 1* [13]

Another example is the construct of the third movement (*Scherzo*) of Beethoven's *Sonata, Op. 28.* This movement displays an A-B-A construction with repeated binary and ternary subdivisions. Thus, the movement have an analogues model to the Sierpinski triangle.

Figure 14. A section of the third movement of Beethoven's *Sonata, Op. 28*, showing the repeated binary and ternary subdivisions. [14]

B. Fractals Analogues in Music Through Pink Noise and Note Reductions

As a result of a growing interest in the phenomenon of fractals, now exist a repertoire algorithmic composition based on fractal generating equations. These fractal algorithms are applied to music in such a way. It is applied to the pitch, dynamics, duration, and other parameters to determine the compositional process.

The automatic music composing of such structures has made it possible to discover in music that is called 1/f noise. 1/f noise also known as pink noise, is a signal or process with a frequency spectrum such that the power spectral density is inversely proportional to the frequency of the signal. This means that it possesses the property of scaling, inherently contained in a soundwave.

In a research paper by Kenneth J. Hsu and Andrew Hsu, [15] they tried to analyze the power-law relations between two successive intervals in a music composition. Kenneth and Andreas Hsu were also interested in =to the problem of reducing music in order to find the smallest self-similar section possible.

The theoretical basis for the postulate that a music score could be reduced into 1/2, 1/4, 1/8, and more is the scale independency or self-similarity of a fractal landscape. The British Isles, as an analogy, as defined by their coastlines, look pretty much the same whether the map is printed on a scale of 1:1,000,000, 1/50,000, or 1:25,000. Music with self-similarity should be susceptible to analogous scale reduction.

By using Bach's *Invention No. 1, BW 772*, they reduced several notes into fractals, reducing them to 1/2, 1/4, 1/8, and so on.

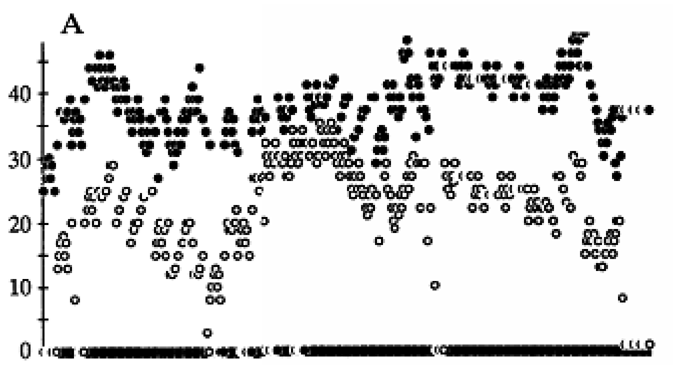

Figure 15. Digitized score of Bach's *Invention No. 1 BW 772*. ○, right hand; ●, left hand. [15]

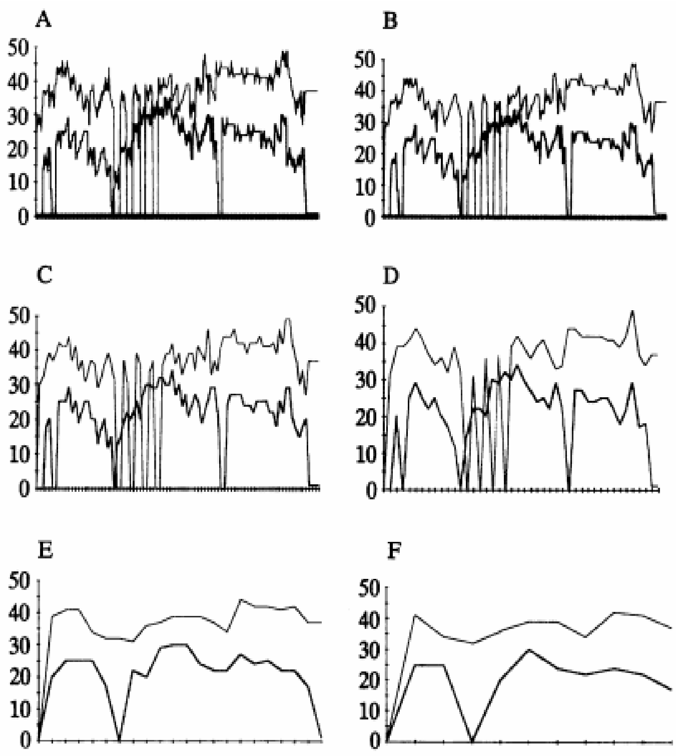

Figure 16. Fractal reductions of Bach's *Invention No. 1, BW 772*. (A) Original. (B) The 1/2, 1/4, 1/8, 1/16, 1/32 reductions of the score, respectively. [15]

By looking into Figure 16., the music shows self-similarity. The music also shows fractal geometry. To someone who may not know and hear music a lot, the 1/2 and 1/4 versions of the score have the impression of feeling like Bach's work. The 1/8, 1/16, and 1/32 versions may feel different in comparison, but the overall structure between those versions are similar and preserved. The final reduction, which is a reduction to only a 1/64 of the score, gives only three notes. Thus, they led a conclusion that those three notes are the foundation upon which the whole composition was built.

V. Conclusion

Fractals could just not be seen but they could also be heard. A fractal may not be obvious in music. The fractal aspect of music is one the many fascinating thing to learn from fractals. Noted by Mandelbrot himself, every natural thing may hold the basis of fractal. But, since we cannot see them directly, it raises a curiosity to find an answer and to find the fundamental essence of fractal and musical beauty.

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STATEMENT

With this statement, I declare that this paper is a product of my work, not an adaptation, a translation of other person's work, nor formed by a result of plagiarizing.

Bandung, 6 December 2019

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