Graph Application in The Strategy of Solving 2048 Tile Game

Harry Setiawan Hamjaya and 13516079
Program Studi Teknik Informatika
Sekolah Teknik Elektro dan Informatika
Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia
harryhamjaya98@gmail.com

***Abstract*—2048 numbered tile game is a game which played on a 4×4 grid, with a number on each tile whether it is filled by a number or even it is left empty. The main goal of this tile game is to get a tile in the given 4×4 grid become 2048 by sliding the numbered tile on a grid to combine them to create it. When the game is launched at the first time, people around the world have spent much time to create a 2048 tile in a single game. Besides the game was very addictive, it also serves us with the interesting fact of mathematics application in real life. One of the Mathematics application in this game is the number theory which I would explain about the number turn we need to reach the goal, and also the biggest number we could have in the tile, and also with the fact I explained before I could derive the most optimal solution in reaching the goal in 2048 numbered tile game.**

***Keywords*—2048 numbered tile game, number theory, graph.**

I. Introduction

2048, a tile game which was built by basic multiplication by 2 or some would say it as the game which each tile is made of tile which are powers of 2. In each turn, one of the possible two numbers 2 or 4 would appears on a random empty field. The game is a sliding block game which means that we could play it by sliding the block in any four directions: up, down, left, right.

2048 numbered tile game is a single-player sliding block puzzle game which was developed by *Gabriele Cirulli*, who is a Italian web developer and released on 9 March 2014 as free and open-source software which was originally written in JavaScript and CSS during a weekend. Actually, *Cirulli* created the game in a single weekend only to see whether he could program a game from scratch, and was surprised when his game received over 4 million visitors in less than a week, especially since it was just a weekend project, and also became the talk of the town within people all over the world, spending hours trying to beat the addictive little puzzle game. The game ends when the user does not have any legal moves left.

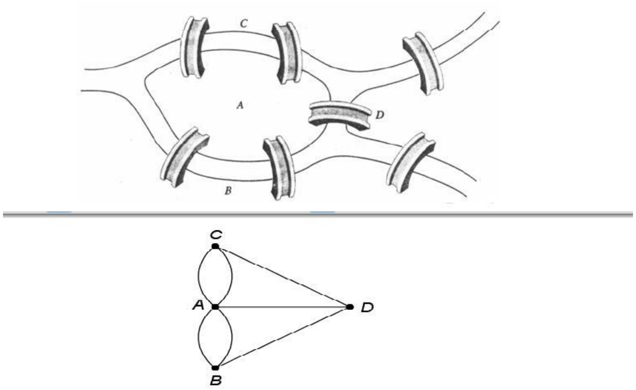
Right at this paper I would like to explain the number theory application in 2048 tile game. Some of them are I would explain what is the largest number you could achieve in the 2048 numbered tile game and I would also analyze what is the minimum number turns do we need to complete the game especially to get tile with number 2048, and with all the fact I had from the observation in number theory, I would derive the best strategy to reach the goal which is to get the 2048 numbered tile before we couldn't have any turn again which mean the game is over.

Picture I: Illustration 2048 tile game
Source: http://gabrielecirulli.github.io/2048/ (access 1 December 2017, 3:58 PM)

II. BASIC THEORY

A. Graph Defenition

According to ref[1], graph is a subject that has been found long time ago, but still applicable until nowdays. Graph is used to represent the discrete objects and the relation between the objects. The visual representation of the graph is by represent the discrete objects as a node, while the relation between them as a line. Graph is very useful nowdays, many games use the graph as represent the gameplay of their games.



Picture II: Illustration Königsberg Bridge[1]

Mathematically, graph is defined as the pair of set (V,E) where:

- V stands for nonempty set of the *Vertices*
- E stands for set of the *Edges*

Graph also could be written as G=(V,E).

Vertices of the graph could be numbered by the alphabet such a, b, c, …, v, w, …, and also by the natural numbers such 1,2,3, …, or the combination between both of them. On the other hand if e is an edge that connect v_a and v_b then we can write

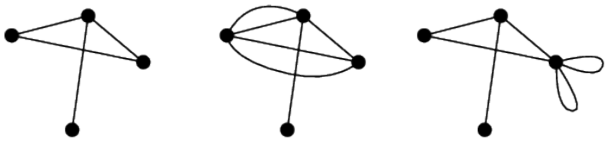
$$e = (v_a, v_b)$$

B. Graph Types

Grouping graph could be represent in many ways. It could be according to double and bracelet edge, and the direction on each edge.

Grouping graph according to the bracelet and double edge could be divided into two groups:

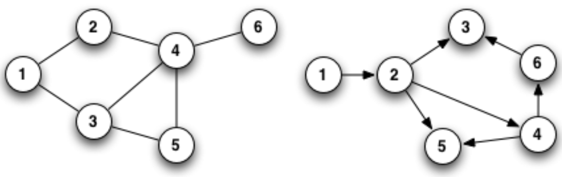
1. *Simple Graph*: graph that doesn't have any bracelet and double edge.
2. *Unsimple Graph*: graph that has at least one of bracelet edge and double edge.
3.



Picture III: Simple and Unsimple Graph [4]

Grouping graph according to the direction on each edge could be divided into two groups:

1. *Direcred Graph*: graph which each edge has a direction where it means it will have each *initial vertex* and *terminal vertex* which is absolutely unique.
2. *Undirected Graph*: graph which don't have any directionon each edge.



Picture III: Directed and Undirected Graph [5]

C. Fundamental Terminology

1. Adjacent

Two vertexes on graph is called adjacent if both of them is connected directedly by a single edge. On other word, V_a is adjacent with V_b if (V_a, V_b) is an edge on graph G.

2. Incident

For any edge on graph $e = (V_a, V_b)$, edge e is called incident with vertex V_a and V_b.

3. Isolated Vertex

Isolated vertex is a vertex which doesn't have any edge which is incident with it.

4. Null Graph

Null graph is a graph that doesn't have any edge.

5. Degree

Degree of a vertex is the number of edge which is incident with the vertex.

There is a useful lemma in degree called shake hands lemma:

The sum of all degree of all vertex in a single graph is always even, since it is twice the number of the edge on the graph. In mathematical expression if we have $G = (V, E)$ then we will have:

$$\sum_{v \epsilon V} d(v) = 2|E|$$

Where $d(v)$ means degree of vertex v.

6. Path

The path length of a graph G is the number of edge which will be traversed from the initial vertex v_0 until the destination vertex v_n which will be alternately changed from vertex to the edge and the edge to the vertex which we could form: $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ where $e_1 = (v_0, v_1), e_2 = (v_1, v_2), \ldots, e_n = (v_{n-1}, V_n)$

7. Cycle/circuit

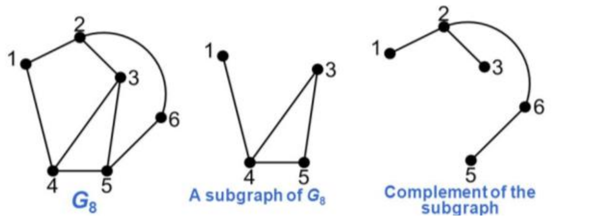
Cycle is a path of a graph that start and finish the path in the same vertex.

8. Connected

Connected graph is a graph which every pair of vertex u and v in set of vertex V have at least one path from vertex u and v. If this property isn't full filled then the graph isn't connected.

9. Subgraph and Complement Graph

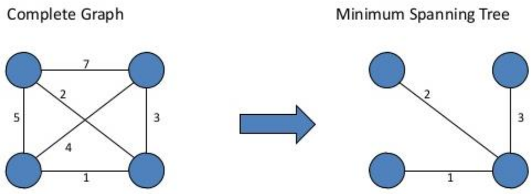
Let $G = (V, E)$ is a graph. We can say that $G_1 = (V_1, E_1)$ is a subgraph of G if $V_1 \subseteq V$ and $E_1 \subseteq E$. While the complement of subgraph G_1 of G is $G_2 = (V_2, E_2)$ such that $E_2 = E - E_1$ and $V_2$are the set of the vertices which are incident with the edges E_2.



Picture IV: Subgraph and Complement Subgraph[6]

10. Spanning Subgraph

Subgraph $G_1 = (V_1, E_1)$ from $G = (V, E)$ is called spanning subgraph is $(V_1 = V)$ where G_1 contain all vertex of G.



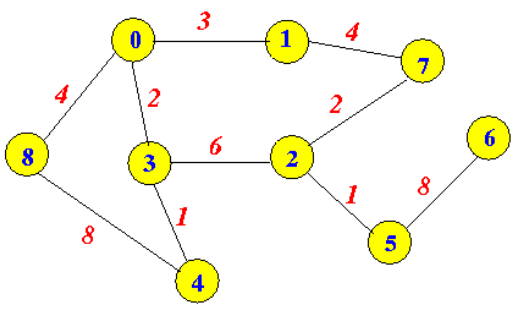
Picture V: Spanning Subgraph[7]

11. Cut-Set

Cut-set of connected graph G is the set edge which are deleted from graph G and make G is not connected anymore.

12. Weighted Graph

Weighted graph is a graph which each edge given a weight such that there is value on each edge.



Picture VI: Weighted Graph[8]

D. Fundamental facts in 2048 tile game

Before I start to tell about how to solve the 2048 tile game at first, we have to know about the fundamental thing in 2048 tile game such the maximum numbered tile we can have in the 2048 tile game, also the minimum numbered way we can have to finish the game it means that the number turn we can have such that we can have tile with number 2048, and last after the facts have been presented I will start explain how to finish the game with strategy and not ending game with game over.

1. Power of 2.

The first observation that al tiles are power of 2. It is obvious since by the mathematical induction we will show that any stage in the game, all tiles are powers of 2. To prove it we could start with:

1) At first, all tiles are powers of 2.

2) Assuming that at some stage, all tiles are powers of 2, then we assume that the next stage will have the powers of 2.

The first point is straightforward since both 2 and 4 are the powers of 2 which is $2 = 2^1$ and $4 = 2^2$.

For the next point / second point, we can have several moves:

I. We could slide it into an empty spot, such that no tiles are combined with another.

II. We could combine two or more existing tiles with another.

III. A combination between I and II.

In both three cases, a new tile will be randomly generated and appeared in the game, which will be either 2 or 4 and by default and proved in the first base case we will always have the power of 2. For the combined one case, we can have that the tiles could be combined if and only if they have the same number on the tiles and they are adjacent then we can have assumption 2^{k-1} are located adjacent each other than if we slide and add it we will have:

$$2^{k-1} + 2^{k-1} = 2 * 2^{k-1} = 2^k$$

Since we have all the power of 2 then it could always be solved in recursive hypothesis. Since the 2^k is also the power of 2 then we have done with the hypothesis and the induction with the base case and the recursive case.

Picture VII: Power of 2[3]

III. SOLVING STRATEGY 2048 TILE GAME

A. The Maximal Tile

From the previous fact and could be easily proved by the recursive and mathematical induction ways we could have that all tiles in the game will always be the power of 2 or in the mathematical form 2^n, so that the maximal numbered tile that could appear will also be in that form.

Let us assume the maximal numbered tile in the 2048 game is 2^r where $r \in N$. To create and have this tile, at least 2 tiles in form of 2^{k-1} are required where they need to be adjacent to each other, and recursively we will have such the 2^{r-1} tile will need the at least 2^{r-2} tiles with both of them are adjacent and then we could combine it, and so on. The fact that we couldn't ignore is the number of tiles is limited by the space on the grid which is 16 spaces, which means at least we will have $r = 16$. Since

$$2^{16} = 65536$$

Is the maximal number we could have if the last tile which make it game over is numbered 2 at the end such we could have the situation in this picture below.

Picture VIII: The Maximal Tile[10]

But If we have numbered 4 appear in the last tile then of course we are very lucky that we could combine it again and we could reach other maximal number that are greater than 2^{16}, which is $2^{17} = 131072$.

The other thing which we could make maximum is the score which of course we can see that the score could be greater if we could combine each of them. Such if we combine 2 and 2 then we will have a new tile let say it was 4 hen our score will be increase for 4 points. For the demo please take a look at the picture and from the demo and the explanation we can derive a new equation that will explain total score that we could have it we want to count our high score that could be achieve by playing this game:

Picture IX: The Maximal Tile[10]

From the picture we could see that 16 could be made by $8 + 8$ while 8 could be made by $4 + 4$ and 4 could be made by $2 + 2$. From here we could see that the total score we could achieve to make 16 is:

$$1 \times 16 + 2 \times 8 + 4 \times 4 = 3 \times 16 = 48$$

From this we can see that if we want to count the score for making tile with number 16 we will have 3×16 or we can say that for each level we could have 2^n for each level but we couldn't count the last level because it is generated and appeared randomly in the grid 4×4 and since we know the fact that: $16 = 2^4$

We can do some mathematical induction to prove that for a tile 2^n, the score we could make:

$$(n-1) \times 2^n$$

Which we can prove such:

Let say for all $k - 1 \in N$ is true for this than we will prove for k. For making tile in 2^k, obvious that we will need $2^{k-1} + 2^{k-1}$ which mean of course we have to add their score before too:

$$2 \times \left((k-2) * 2^{k-1}\right) + 2^k = (k-1) \times 2^k$$

Which make our hypothesis true and by mathematical induction, we have it's true.

B. The Minimum Turn

The game would be said finished or we could say we won the game if and only if we could create the 2048 tile (even though you could still play to get your best high score after you achieve the 2048 tile).

Since there are two types of tiles that are generated which is 2 with the probability 0.9 and 4 with the probability 0.1, Of course we couldn't give a specific and exact numbers for number of turn to finish the game by achieve the 2048 tile however we could of course give a range of turn to this possibilities through the calculation between the worst case and the best case.

Let we divide it into two cases such that the first one is the best case and the second could be worst case:

I. Assume the best scenario is we will always get the number of 4 for every tiles combination and always assume that the tile that are generated and appeared always *adjacent* (this means we don't need some move to make it adjacent again), when we are merging 2 tiles into 1. Before making a 2048 tile of course we need to use 2048×4^{-1} tiles which are equal to 512 tiles. But we need to aware that we have the 2 bonus at the first that means we need $512 - 2 = 510$ tiles that mean we need 510 turn to move. Another thing we have to consider is the last tile that we have must be combined to make 8, then 8s into 16, then 16s into 32, then 32s into 64, then 64s into 128, then 128s into 256, then 256s into 512, then 512s into 1024, and last 1024s into 2048.

From this analysis, we could get:

$$T_{min} = 512 - 2 + 9 = 519$$

Thus, we will have at least 519 turns to solve 2048 tile game with the grid that is generated is always 4 and because it is best case then the tiles will always be adjacent so we didn't need to move 1 turn just in case to make it adjacent.

II. Now we will consider for the worst case, let say the worst case that we could achieve condition will be such all the tile that are generated randomly is always be the number 2 tile, then we will have of course more moves than the first one but we also have to consider that the generated number will pop out no in the adjacent with the other number this means it is really worst case. Let us use another trick to count it, by using a recursive formula. To make a tile with numbered 4 and using only 2 tiles 2 will need 2 turns, to make tile with numbered 8 will approximately need at most 5 turns, and so on

From the fact we have, we could rewrite it in:

$$T_{min}(2) = 2$$
$$T_{min}(3) = 5$$
$$T_{min}(4) = 8$$
$$T_{min}(n) = 2 \times T_{min}(n-1) + 1$$

We could derive the formula since to make 2^n we could make it by combining 2^{n-1} that means it equal to find 2^{n-1} turn as twice and plus one since we need to combine them to make the 2^n

Form here we could derive $T_{min}(n) = 2^n - 2^{n-2} - 1$ for all $n \geq 2$. From this fact we could have:

$$T_{min}(2048) = 1535$$

Thus, we will have at least 1535 turns to solve 2048 tile game with the grid that is generated is always 2, since we want to find the best range then we assume that the tile that are generated randomly as always is not adjacent to the tiles which are similar to it and it means we need to make it become adjacent first and then making it to be combined with each other

From the analysis above we could consider that the minimum number of turns required to win the game and achieve the main goal which is 2048 tile game, and assume that we could play as perfect and as well as possible, it would be in range [519,1535].

C. Graph Application in 2048 tile game

So far we have known all facts in 2048 tile game, I think this the time we need to explain the graph path that you should get to win the game.

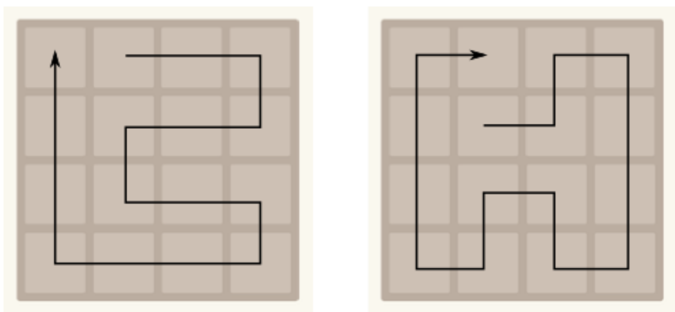
It all has to do with the decompositions of these numbers as sums of powers of 2. To have the tiles fitting in the bounded 4×4 grid, we need to find decomposition as efficient as possible, in the sense that we want to write the sum of tile numbers with the smallest possible number of powers of 2.

It's easy to see that, in any most efficient decomposition, powers of 2 must appear only once. Indeed, otherwise, we can just combine 2 identical powers of 2 into their sum (note that it must be adjacent to each other, otherwise we can't combine it), hence yielding the same sum with one less tile. For more evident we could see the picture below.

Picture X: The Configuration[3]

In an ideal configuration, the tiles must not only be adjacent to similar numbered tile but also with the higher numbered tile so formed must be adjacent to another tile of the same number and so on with the highest numbered tile being left in the end, which if we want to describe it using graph we would have each grid tiles as the vertex and the edge would be all the sides of the tiles which are adjacent to each other.

From the analysis, we need to perform such a snake line that make all the tile in the grid connected each other and could be combined easily. Therefore, some of the people who want to brick the high score prefer to make such the tile in the grid has a monotonicity increasing values such that it can be easy to combine, let us take an example if we have a large value on r but both 2^r tiles aren't located adjacent to each other. It would be harder to make another 2^r tile, not only because we would have to spend much time, but we also would have to think about how to make the tiles with the limited space since if we can't have any blank space for any new generated number, we would lose the game and the game is over. That's why some of the player use to make it such a snake line configuration where it could be easily combined if it is the same rather than breaking the chain between the bigger one number tiles.



Picture XI: Some of Optimal Path for Best Strategy[3]

However, in practice of course doing and placing such one of the best and optimal configuration isn't as easy as well the explanation it is because the game has a randomize function that will make we can't predicted which configuration could we make due to it is very-very random. Since the tile will not always generated and appeared as our wishes. There will be three cases that could happen:

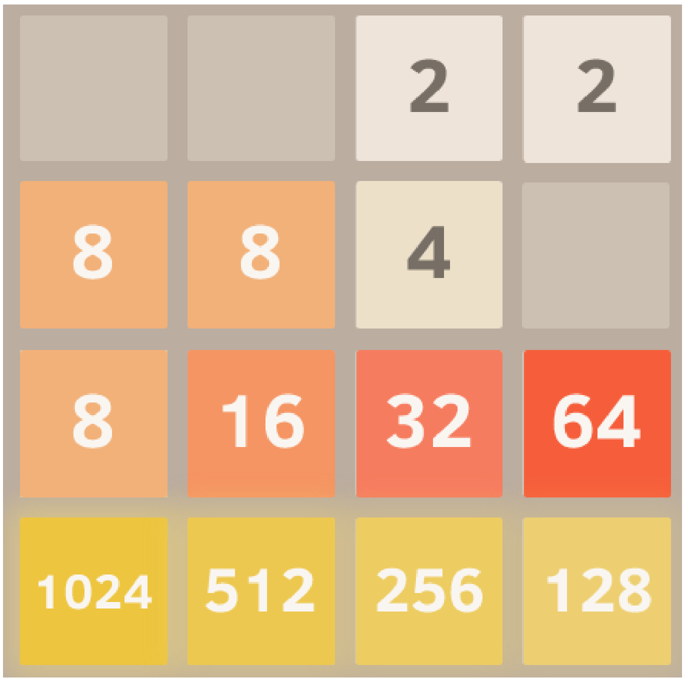
1. Best Case: it happens that the generated random number appear in the place where we would like them in that position

2. Average Case: it happens when it is literally random

3. Worst Case: it happens that the generated random number appear in the place where we wouldn't like them in that position

C. Graph Isomorphism Application in 2048 tile game

To get such position we could always to make it such the pascal triangle form with the highest number at the corner such the below picture. We try to make our target $2^{11} = 2048$ tile can be achieved in the bottom such the picture below.



Picture XI: The Pascal Tile in The Grid[9]

In order to achieve this state, one of the possible directional (up, down, left, right) move must be restricted. So that we can make the chain of our possibilities in the snake line configuration could be maintained as well as possible. But if it is necessary to use the restricted move, we can counter it as fast as possible with the opposite directional moves.

Since the graph is isomorphic make us choose one of the directed moves easier and we didn't need to afraid about choosing the wrong restricted moves since all of them are equal since it isomorphic to their rotation.

V. Conclusion

The conclusion for the best strategy is by avoiding one move we can make the tile as a snake line configuration such that the chain isn't broken. Well such the assumption before we have that the tiles as the vertexes and the adjacent sides as the edges we could make the biggest value vertex/ tile at the one side of the grid and still using the non-restricted move to solve the tile game. But the isomorphic graph configuration made us easily to choose the restricted moves, without any special condition and special requirements.

VII. Closing

Gratitude be upheld to the one almighty god for the grace and the inclusion in making this paper so that it could be finished as well. The author's grateful to Mr. Rinaldi Munir, Mr. Judhi Santoso, and Mrs. Harlili as our lecturer in Discrete Mathematics Course for the time and the knowledge which are shared with all of us. Also grateful for all of family and friends for help and support until now.

References

[1] Munir, Rinaldi. 2006 "Diktat Kuliah IF 2120 Matematika Diskrit.", Institut Teknologi Bandung : Bandung
[2] Goel, Bhargavi, 2017 "Mathematical Analysis of 2048, The Game", access in https://www.ripublication.com/aama17/aamav12n1_01.pdf at 1 December 2017
[3] http://www.science4all.org/article/2048-game/ access at 1 December 2017
[4] http://mathworld.wolfram.com/SimpleGraph.html access at 2 December 2017
[5] http://think-like-a-git.net/sections/graph-theory/directed-versus-undirected-graphs.html access at 2 December 2017
[6] http://slideplayer.com/slide/2433835/ access at 2 December 2017
[7] https://www.slideshare.net/ganith2k13/graph-theory-26101317 access at 2 December 2017
[8] http://www.mathcs.emory.edu/~cheung/Courses/171/Syllabus/11-Graph/weighted.html access at 2 December 2017
[9] http://zillionthoughts.com/2048-simple-game-mathematics/ access at 2 December 2017
[10] https://www.youtube.com/watch?v=OO4tA5i7X9g access at 2 December 2017

Pernyataan

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

Bandung, 3 Desember 2017

Harry Setiawan Hamjaya/13516079