The Role of Combinatorics in Hearthstone

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***Abstract*—This writing is made in order to expose how discrete mathematics, combinatorics specifically, affects a lot of aspect of a turn based card game, for example, Hearthstone. Like all other card games, it involves drawing cards, playing cards, etc. Combinatorics and chances play a big role in card games. One of the examples where they play is in the drawing mechanics.**

***Keywords*—Combinatorics, Card Games, Luck Factor, Sequence**

I. Introduction

Card games have existed in human history ever since the 14th century[1]. There are many forms of it during those years, each has its different uniqueness and of course different types of cards. The playing card that is the most well-known by us is probably the French playing card, which consists of four cards with four suits and two colors, each from ace to ten, and also four jacks, queens, and kings (and sometimes jokers, too). Card games slowly thrive through the ages, some stayed until now, some are no longer touched, and new types of card games appear to fit the new era.

On the 1990s, a new type of card game made an appearance – a collectible card game. A man named Richard Garfield, who were a graduate who were working for a PhD in combinatorial mathematics created a card game called Magic: the Gathering[2]. It has a system different from all of its other predecessors, you do not have all the cards until you have it. There lies the concept of packs where you can open it and get cards that you can use for playing. This concept went through the times quite well and until now there are many others that holds this idea.

One of those who holds the concept of a collectible card game is Hearthstone. Hearthstone is a game created and developed by Blizzard, which is a card game that is based on the Warcraft lore. A lot of the cards came from the Warcraft world, or based on it. It was first announced on March 2013, and then released on BlizzCon 2013 on November on the same year[3].

The game starts with two players, both on 30 health. In the standard or regular mode, each player basically has to deal damage to the other player using the cards on each players' deck. This involves drawing cards, and playing them – summoning minions on board, or casting spells. Minions by themselves have stats bound in them – attack and health, and tremendously varying effects in each different card, or none at all. Spells also have a lot of different effects when you play it, from dealing damage, summoning minions, drawing cards, etc.

Each of the features said above are possible to be studied from the perspective of combinatorics. We can count how many arrangement or combinations of drawing sequences that could possibly happen. Usually in practice this is combined with chances, of which then we can calculate what are the chances to draw certain cards, or what are the chances to get a bad effect from a certain card effect, and so on.

The authors' purpose of making this paper is to expose on how combinatorics and chances play a big role in this type of card games. This paper will not only expose on the card drawing mechanics, but also on how the effects of some cards in the game that touches combinatorics – showing how much a card games' outcome may vary.

II. Theory on Combinatorics[4]

A. Definition and Examples

Combinatorics are a branch in mathematics that studies how objects are arranged. From combinatorics we can tell how many possibilities that we could achieve from a set of objects. Examples would be to count how much possible moves that could happen in a chess game (of course this is a difficult task), or how many possible combinations that could happen if we take two coins from a huge pile of coins with different types.

B. Basic Rule in Calculating Combinatorics

In combinatorics, most of the time we do need to count all the possibilities, meaning the sum of the amount of the possible combinations. And therefore, there are two type of rules that we can use to achieve so.

1. *Rule of product*

   If an experiment, let us say experiment 1, yields p number of possible outcomes, and another experiment called experiment 2 has q number of possible outcomes. When both experiments are done (experiment 1 <u>and</u> experiment 2), there are p x q number of possible outcomes that is generated by the two experiments.

2. *Rule of sum*

   If experiment 1 yields p number of possible outcomes, and experiment 2 has q number of possible outcomes, when only one experiment can be done (experiment 1 <u>or</u> experiment 2), there are $p + q$ number of possible outcomes that is generated by the two experiments.

C. Rule Expansion

Rule of product and rule of sum both can be expanded and therefore could include more than just two experiments. If n number of experiments each have p_1, p_2, …, p_n. where pi does not depend on its previous ones, then the number of possible outcomes are:

(a) $p_1 \times p_2 \times \ldots \times p_n$ for the rule of product, and
(b) $p_1 + p_2 + \ldots + p_n$ for the rule of sum.

Rule expansion is usually used when you need to count more than just two occurrences of which we can take the number of possible outcomes. For example, we can count the number of possible combination of a string with a length of ten, consisting of zeros and ones, of which we can use the expansion of the rule of product.

D. The Inclusion-Exclusion Principle

There are cases in combinatorics where you need to combine two numbers of possibilities, or exclude certain cases (usually cases of which you need to exclude are those which are calculated twice, even though both are the same). For example, when you try to add the number of possible outcome of a binary string that ends with '11' and a string that begins with '11', there are cases of which both already cover. Those cases are the ones that we try to exclude, while the others we try to include.

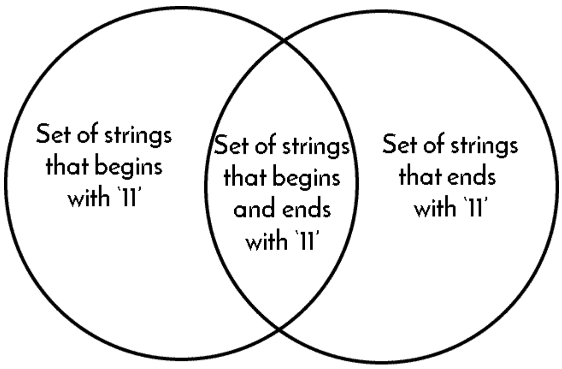

Figure 1. A Venn diagram representation of the said example

E. Permutation

Permutation is the number of possible combinations in arranging objects. It uses the rule of product as a base, or it could be said as permutation is an application of the rule of product. Let us take an example of a number of object n, then the first sequence is chosen from n objects, then n - 1 objects, then n – 2 objects, and so on until there is only one object left. According to the rule of product, the permutation of n number of objects is

$$n(n-1)(n-2)\ldots(2)(1) = n!$$

Equation 1. Permutation of an n number of objects

Although, it is different yet similar for cases where you try to arrange a r number of objects, picked from n number of objects. According to the rule of product, when you are trying to arrange r number of objects, choosing from an n number of objects, there is

$$n(n-1)(n-2)\ldots(n-(r-1))$$

Equation 2. Permutation of an r number of objects, selected from an n number of objects

number of different arrangements that can be done. It is also called *r-permutation*, usually symbolized as P(n, r), which is

$$P(n,r) = n(n-1)(n-2)\ldots(n-(r-1)) = \frac{n!}{(n-r)!}$$

Equation 3. The permutation formula. Alternate form of equation 2.

When $r = n$, then we can use equation 1 to count the number of possible arrangement of n number of objects, usually written as

$$P(n,n) = n(n-1)(n-2)\ldots(n-(n-1)) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Equation 4. Another definition of equation 1.

F. Combination

Combination is rather similar to permutation, but it is different in a way that the same sequence of appearance is not counted as an extra possibility. So, if for example we have *abc,* then *bca* and *cba* would be counted as the same – it counts as one single possibility. A combination of r number of elements from n number of elements is the amount of unsorted selection of r elements from n number of elements. It is usually written as

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

This could also be proven by saying that P(n,r) is basically C(n,r) that is sorted with P(r,r) possibilities, or written like so:

$$P(n,r) = C(n,r)P(r,r)$$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!}{r!(n-r!)}$$

- Interpretation of Combination

  The matter of combination can be interpreted as so:
  1. Combination C(n,r) to count how many subsets that consists of r elements that is formed from n elements. Usually in this case, the existing subsets that are repeated but is served in a different order are considered the same sets, and therefore considered as one.
  2. Combination C(n,r) to say that selecting a certain r elements from n elements, can be said as selecting those objects without considering whether the selection is sorted or not.

G. Generalized Permutation and Combination

There are cases where the objects within n are the same. For example, we can take a look on n objects, but there are n_1 objects that has the initial letter A, n_2 objects with the initial letter B, and so on until n_k. From there we can tell that $n_1 + n_2 + \ldots + n_k = n$. If we are to put these n objects into n number of empty spaces, then there would be

$$P(n; n_1, n_2, \ldots, n_k) = \frac{P(n,n)}{n_1!n_2!\ldots n_k!} = \frac{n!}{n_1!n_2!\ldots n_k!}$$

Same goes for combination, although it is derived differently. It is derived like so

$$C(n; n_1, n_2, \ldots, n_k) = C(n, n_1)\, C(n - n_1, n_2)\, C(n - n_1 - n_2, n_3) \ldots C(n - n_1 - n_2 - \ldots - n_{k-1}, n_k)$$

$$= \frac{n!}{n_1!(n-n_1)!} \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_k)!} \ldots \frac{(n-n_1-n_2-\ldots-n_{k-1})!}{n_k!(n-n_1-n_2-\ldots-n_{k-1}-n_k)!}$$

$$= \frac{n!}{n_1!n_2!n_3!..n_k}$$

Source: Munir, Rinaldi. 2003. Matematika Diskrit.

This equation is called *generalized combination*. Both the generalized permutation and combination can be calculated with the same formula, as they both are the same.

H. Combination with Repetition

If we were to try to take *r* objects from *n* objects, where each *r* can only be filled with one, then we can say that the number of combinations that can be created are C(n,r). But, it is different when each *r* can be filled with more than one *n*. Then, there would be C(n + r – 1, r) number of combinations that is possible. C(n + r – 1, r) is the number of combinations that allows a repetition of objects.

III. COMBINATORICS' ROLE IN HEARTHSTONE

A. Hearthstone's Basic Mechanics – an Introduction

When you begin a game in Hearthstone, as said before each player starts with 30 health, and starts with a deck of 30 cards. At the beginning of the game, you are given 3 (or 4 if going second) random cards from your deck, and you can either choose to redraw any of the cards once, or you can keep it. Then from there you start with 1 mana. Each turn your mana grows by 1 (if not added or removed by card effects). This mana can be used in order to play your cards.

There are several features or mechanics that involves a game of chance and luck. The most basic one is card drawing, for example. Each turn, a player draws a card, and from that if we knew the contents of the deck, we can tell how many possible arrangements of drawing sequence that could happen the following turn. Then, there are card effects. Some minion cards have an effect, an effect when played called battlecry, an effect when the minion dies called deathrattle, etc. We are going to take a look on the effects themselves, not on how the effect is triggered. An example of a card effect would be like so

Source: https://hearthstone.gamepedia.com/File:Novice_Engineer(435)_Gold.png

It starts with a battlecry, means that the effect will trigger upon the card being played, which is to draw a card.

There are a lot more effects than to just drawing a card, but it is not going to be written here. The effects are going to be shown as the author writes, along with its combinatorial role.

B. Combinatorics on the Start of the Game

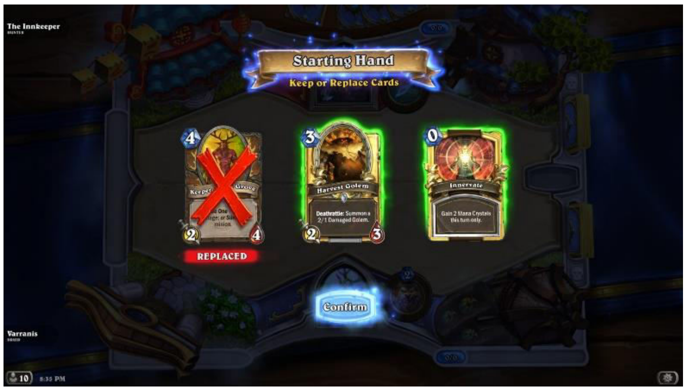

Figure 2. The start of a standard Hearthstone game

During the beginning of the game, you start with 3 or 4 cards (a starting hand) that you can choose to either redraw any of the cards or to keep it. Depending on the playstyle and the match up, you usually want to keep or redraw certain cards because it is either too slow, or too weak against a certain class. Let us take an example, that we have a deck of 30 cards, each type of cards consists of 2 (the maximum number of a card to be in a deck, if not a legendary class card), and we are going first. So, it is possible to draw 2 same cards at the beginning. The number of combinations that could happen during this event would be

$$C(30,3) = \frac{30!}{(30-3)! \cdot 3!} = 5 \cdot 29 \cdot 28 = 4060$$

We are using combination because we are omitting the order of which the card is drawn. The duplicates of each card are not accounted here because we are looking at the general case first. But, if you were trying to count the number of combination that could occur if you were to try to draw a duplicate, then it is counted like so

$$C(15,1) \cdot C(28,1) = 15 \cdot \frac{28!}{(28-1)! \cdot 1!} = 28 \cdot 15 = 420$$

There are 15 different cards, and there are two of each card in the deck. If we are trying to draw at least a duplicate, we would first take 1 from 15 different cards, that we multiply it with when we take one card from a pool of 28 cards (because the two we already took).

Now let us take a different case. We currently have a deck with no duplicates, and there are two core cards that lies within the deck. The number of combinations that could happen in order to immediately draw those two cards on the starting hand would be like so

$$C(28,1) = \frac{28!}{(28-1)! \cdot 1!} = 28$$

We already took two cards away, and there is only 28 left, with 1 card slot left in the starting hand. Compare the number of combinations to draw that two cards immediately with the total combinations, which is 4060. The chance for you to draw these two cards immediately is actually really slim.

The order of which the cards are shuffled can also be calculated with combinatorics. Assume you have a deck with duplicates (maximum two cards each), and also assume that you are going second (post-redrawing). The number of possible arrangements that could be done would be like so

$$P(26; 2,2,\ldots,2) = \frac{26!}{13(2!)} = 25! = 15511210043330985984000000$$

There would be that many ways that your deck can be shuffled, and that makes card games that requires you to draw cards are unique almost every time, since it is extremely unlikely to draw with the same sequence as the previous games, or other people's games.

C. Combinatorics in Drawing Cards

Like in other card games, Hearthstone also involves drawing cards – taking a card from the top of your deck and putting it onto your hand. The number of sequence that could happen during this could be extremely various. Let us take an example on a deck of 30 with no duplicates, and assume that we are going second (which means we have 4 cards as our starting hand). From the beginning of the game until the deck runs out, there would be a large number of possible sequences in drawing the cards.

$$P(26,26) = 26! = 403291461126605635584000000$$

Let us say that in our deck, we have 3 of each card having a unique mana cost (3 one mana cards, 3 two mana cards, and so on). Let us say that we kept 3 one mana cards as our starting hand. The possible number of sequences that we take 2 mana cards, 3 mana cards, and so on in a consecutive manner until we get 10 mana cards would be like so

$$P(3,1)^{10} = 3^{10} = 59049$$

As it may seem like the number is quite big, if we compare it to the total possible drawing sequences that could happen it is actually really less likely for us to draw in such a sequence, or to draw in any sequence without taking a card with the same mana.

D. Combinatorics in Card Effects

As said previously, Hearthstone cards sometimes has effects written on it. There are so many cards in Hearthstone with random effects that the author may not be able to expose them all onto this paper. Let us start with an effect, called "Discover".

1. Discover

Source: https://hearthstone.gamepedia.com/

Jeweled Scarab – Discover a 3-cost card.
Source: https://tempostorm.com/articles/hearthstone-pros-react-to-the-league-of-explorers

Discover is basically an effect that allows you to pick one out of three cards and put in onto your hand. It either immediately put in onto your hand, or it could do other things to the card you pick (immediately cast the spell you pick, or reduce its cost, etc.). The cards could either be picked from the class you are using, or it can also be from other classes. Let us take an example on Primordial Glyph.

What this card does is that it takes three cards out of a pool of spells that is available depending on the caster's class. For example, there are 35 spells that are currently available as a mage class. Primordial Glyph basically takes three cards and gives you a choice to choose which cards do you want. The number of combinations that is possible is

$$C(35,3) = \frac{35!}{(35-3)!3!} = 35 \cdot 17 \cdot 11 = 6545$$

6545 is the number of combinations possible from a pool of mage spells only. Let us try a larger pool of spells, for example with Tortollan Primalist.

This card does not take spells from the player's class, but rather all existing spells in Hearthstone, which makes a lot more of possible combinations. If we calculate it, the number of possible combinations that could possibly happen is

$$C(366,3) = \frac{366!}{(366-3)!3!} = \frac{366 \cdot 365 \cdot 364}{6} = 8104460$$

As there are 366 spells in Hearthstone, the total from all discoverable spells.

2. *Card/Deck Stealing*

There are also cards that allows you to steal (copy) your opponent's cards, either from their deck or from their hand. These types of cards are usually from the Priest or Rogue class. But the difference is that Priest steals from the enemy's deck or hand, while Rogue takes a completely random card from the opponent's class.

Priest spells that steals cards

Rogue cards that steals cards

The author will specifically focus the Priest cards, as the number of combinations that the Rogue class steals is the same as the number of the existing cards that are in the opponent's class (except for discover). Let us take a look on Thoughtsteal, which effects are "Copy 2 cards in your opponent's deck and add them to your hand". Suppose that there are 20 cards left in the opponent's deck. The number of possible combinations that could possibly happen in a deck with no duplicates are

$$C(20,2) = \frac{20!}{(20-2)!2!} = 10 \cdot 19 = 190$$

If the deck has duplicates, the number of combinations would be reduced greatly. Let us say each card in the deck contains duplicates (which means two of each card, ten different cards). Then the possible combinations would be like so

$$\frac{C(20,2)}{2} = \frac{20!}{(20-2)!2! \cdot 2} = 5 \cdot 19 = 95$$

3. *Discard mechanics*

In Hearthstone, there are cards that allow you to pay the cost of a certain minions' stats by discarding one or more cards. This type of mechanic usually exists within Warlock class. These cards are very risky to play when you have other good cards on hand, and it really depends on your luck. With the help of combinatorics, you could know how many different combinations of discards that could happen when you play a discard effect.

Warlock cards that discards your hand.

Let us take a scenario where we have ten cards on our hands, with a Doomguard on our hand. When we play Doomguard, that means there are 9 possible discard targets that could be discarded. Therefore, the number of combination that could possibly happen is

$$C(9,2) = \frac{9!}{(9-2)!2!} = 9 \cdot 4 = 36$$

If for example we are holding one valuable card in hand, the number of combinations where the valuable card may be discarded would be

$$C(8,1) = \frac{8!}{(8-1)!1!} = 8$$

By chance, there would be an 8/36 (2/9) chance where your valuable card might be discarded. These calculations may help you decide whether you are ready to discard your valuable cards or whether you are ready to take the risk of discarding it.

E. Combinatorics in Matchmaking

There is also something that is called a matchmaking system in Hearthstone. It basically allows two players to have a Hearthstone duel online. As simple as it may seem, it is actually a bigger system than it looks like. Things like skill ratings and win streaks are put onto calculations. But here, we are going to take a look at how many combinations that could possibly happen if we were to take two different players from a pool of online players in Hearthstone.

Assume there are 100000 online players in Hearthstone. The number of combinations of matches that could happen would be

$$C(100000,2) = \frac{100000!}{(100000-2)!2!} = 50000 \cdot 99999 = 4999950000$$

IV. THE TRUE APPLICATIONS OF COMBINATORICS WITHIN HEARTHSTONE

Combinatorics on itself is probably not enough in order to be used within the game of Hearthstone itself when it comes to playing. But, if you combine combinatorics with chances, it will make a lot more sense, as there is a comparison of the tiny fraction of a sample, and a bigger sample.

We can take an example on the card-stealing mechanics that exists on Priest. Let us say that the opponent has a very powerful card that synergizes really well with your current deck or hand. And there is only this one card that you are aiming. Assume the game has went on until there are only twenty cards remaining in your opponent's deck. You have thoughtsteal on your hand, and if you were to play it, the number of combinations that could happen on a deck with no duplicates would be like so as already written above

$$C(20,2) = \frac{20!}{(20-2)!2!} = 10 \cdot 19 = 190$$

And assume you are aiming for this one card. The number of combinations that could happen if at least this key card is taken is

$$C(19,1) = \frac{19!}{(19-1)!1!} = 19$$

By using chances, the chance that you may take this key card from your opponent would be 19/190 (1/10), or 10% chance to take this specific card from your opponent.

Let us take another scenario that may need probability to make a decision. For example, you have a Primordial Glyph and a Kazakus. You currently have 5 mana left, and you are in need of 3 damage in order to win. Kazakus costs 4 mana to play, and it has the choice to create a one mana spell, that could deal 3 damage (it is not guaranteed to get to select damage). While using Primordial Glyph, you have several options, such as Fireball that costs 4 mana and deals 6 damage, Frostbolt which costs 2 mana and deals 3 damage. You also have a hero power that costs 2 mana and could deal 1 extra damage.

Primordial Glyph has the selection pool of 35 existing mage spells, while Kazakus has a pool of 9 possible one mana spells to be selected, and you can select two effects from it.

The number of possible combinations that you can possibly get using Primordial Glyph would be so

$$C(35,3) = \frac{35!}{(35-3)!3!} = 35 \cdot 17 \cdot 11 = 6545$$

While in order to get a Fireball or Frostbolt, the number of possible combinations would be like so

$$2 \cdot C(34,2) = 2 \cdot \left(\frac{34!}{(34-2)!2!}\right) = 34 \cdot 33 = 1122$$

Therefore, if you use the Primordial Glyph, there is a 1122/6545 (around 17%) chance of winning when you use it.

But, if you use Kazakus, the total number of possible one mana potions that can be created would be so

First Selection:

$$C(9,3) = \frac{9!}{(9-3)!3!} = 3 \cdot 4 \cdot 7 = 84$$

Second Selection:

$$C(8,3) = \frac{8!}{(8-3)!3!} = 8 \cdot 7 = 56$$

Total:

$$C(9,3) \cdot C(8,3) = \frac{9!}{(9-3)!3!} \cdot \frac{8!}{(8-3)!3!} = (3 \cdot 4 \cdot 7) \cdot (8 \cdot 7) = 4704$$

It is multiplied like so since you can choose two effects from it, and you cannot have the same effects on the second choice, which is why it is C(8,3). The number of possibilities that could win the game would be

1) First case: getting 3 damage from the first selection immediately.

$$C(8,2) = \frac{8!}{(8-2)!2!} = 4 \cdot 7 = 28$$

2) Second case: getting 3 damage from the second selection.

$$C(7,2) = \frac{7!}{(7-2)!2!} = 7 \cdot 3 = 21$$

And therefore, there would be a 28/84 (33.33%) chance that you would win the game on the first try, but also you get a second chance with a chance of 21/56 (37.5%) on the second one. The average chance that you may win is around 35.42% if you play Kazakus instead of Glyph.

V. CONCLUSION

Despite only just playing a game of Hearthstone, a basic knowledge of combinatorics and probability could lead to decisions where things could lead to victory. Some may oversee this as something insignificant, but it is more significant than some may think. A core part of Hearthstone is decision making, when to make a certain decision. And when you are put onto a situation where chances play, combinatorics and probability could play a big role, like the mentioned case above.

Combinatorics can also be used to see how big or how many combinations that could happen in a game, and how big of a number of possible games that could happen with so many probable events. We can take for example in the drawing sequence or the number of possible arrangements of a deck. It is indeed not a small number. Through combinatorics, we could appreciate a game of chances more, how big the possibilities are.

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PERNYATAAN

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

Bandung, 3 Desember 2017

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