Proving Euler’s Formula in Connected Planar Graph using Mathematical Induction

Dewita Sonya Tarabunga
13515021
Program Studi Teknik Informatika
Sekolah Teknik Elektro dan Informatika
Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia
13515021@std.stei.itb.ac.id
dewitast20@gmail.com

***Abstract*— Generally, in algebraic topology, Euler’s Formula states that for any shape or structure, there is an invariant called Euler characteristic x such that, no matter how the structure is bent, the equation $V-E+F=x$ is always true. More specifically, it is known that Euler characteristic for connected planar graph is 2. There had been many proofs published to prove this formula. In this paper, inductive proof is going to be used to prove Euler’s formula.**

***Keywords*—Euler formula, mathematical induction, planar graph,**

I. INTRODUCTION

According to Oxford Dictionary, mathematics is the abstract science of number, quantity, and space either as abstract concepts (pure mathematics), or as applied to other disciplines such as physics and engineering (applied mathematics). Moreover, we can also found applications of mathematics in daily life. There are many branch of mathematics, but two of our concern in this paper are about mathematical induction and graph theory, also how inductive proof can be used to prove a theorem in graph theory.

Matematical induction is one of a proof technique in discrete mathematics, typically used to establish a given statement that some properties are hold for all natural numbers. But not only natural numbers, mathematical induction can also be used to prove any statements in any well-ordered set, like integers, or any subset of natural numbers that is well-ordered.

Historically, many ancient mathematicians have used mathematical induction technique implicitly in their proofs. Application of inductive proof can be found in Euclid’s proof of infinite primes. This is one of the example that showed that mathematical induction not only can be used for natural numbers. In this case, it is used to prove a statement that works on prime numbers.

Graph theory is a branch of mathematics which concern is about how networks can be translated into mathematical structure called graph and how to measure their properties. Graph theoty first originated from the paper “Seven Bridges of Konigsberg” by Leonhard Euler. In that paper, Euler model the bridges of Konigsberg into a graph and try to solve a problem where someone had to cross all the bridges exactly once and in a continuous sequence. The problem is later known as determining wether some graph is a Eulerian graph.

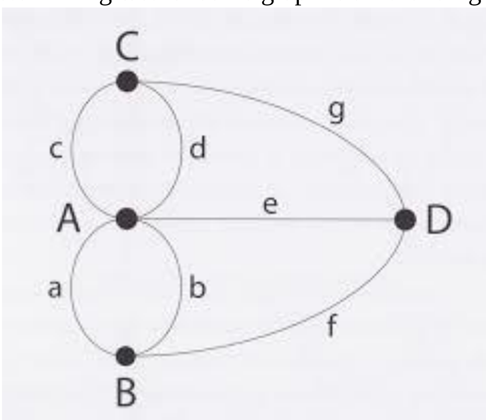

Fig. 2 Graph representation of the Seven Bridges of Konigsberg problem
bigthink.com

Euler had published so many papers and theorems. One of the theorems called Euler’s formula concerns about the topological invariant called Euler characteristic possessed by any geometric structures regardless of how it is bent. In a field of graph theory, Euler’s formula states that the Euler characteristic for connected planar graph is 2.

II. Mathematical induction

Mathematical induction is a method first known to prove propotitions related to natural numbers. Although it is already developed to work not only in natural numbers, but also in any well-ordered set. With induction, we can prove that some propotition holds for all elements in the set with only small numbers of steps. There are many variants of inductive proof. The main principle of this method is that it works like a domino effect.

Fig. 2 Illustration of mathematical induction using dominoes
http://www.chuckgallagher.com/small-choices-matter-the-domino-effect-in-choices/

There are two main steps in mathematical induction. They are base case and inductive case. Below are the base case and inductive case for simple mathematical induction.

1. **Base case**
   As the first step to do inductive proof, we must solve for one base case. That is, we must prove that the propotitions we are about to prove holds for one basis. In a simplest form and most common case, the basis is 1, although the basis is not limited to 1. Formally said, prove that $p(1)$ is true.
2. **Inductive case**
   If the propotition doesn't hold in the base case, we can jump straight to conclusion that the propotition is false. Otherwise, we may continue to the inductive case. First, assume that for some natural number k, then $p(k)$ is true. After that, prove that the propotition also holds for $k+1$. That is, prove that if $p(k)$ is true, then $p(k+1)$ is also true.

With domino effect, two steps above are enough to prove that $p(n)$ is true for any natural number n. $p(k+1)$ is true if $p(k)$ is true. Because $p(1)$ is true, then $p(2)$ is true. Then, because $p(2)$ is true, then $p(3)$ is true, and so on. Now, we have proved that it is holds for any natural numbers.

One of the most important variants of the mathematical induction is strong induction. There is only slight difference in assumption for inductive case in strong induction. If we only assume that $p(k)$ is true in simple induction, in strong induction we assume that the propotition is true for all $i \in \{1, 2, \dots, k\}$. That is, $p(1), p(2), \dots, p(k)$ is true for some natural number k.

As stated before, the basis for the base case is not limited to 1. We can use any basis n_0 to start the mathematical induction.

III. Graph theory

Graph is a simplified abstraction and representation of a network and its connectivity. Graph is a geometrical structure illustrated as a set of discrete objects denoted by points and lines. A point in a graph is called vertice or node, whereas any connection of any two points is called edge. There is a formal definition of graph, that is, graph is an ordered pair of two sets, one is a set of vertex and the other is a set of edges. A graph $G = (V, E)$ where V is a non-empty set of vertex and E is a set of edges (can be empty).

There are already so many applications of graph in daily life, usually involves model of connections, or optimization. Graph is divided into many types, according to its properties. For example, there are simple and unsimple graph, directed and undirected graph, weighted graph and unweighted graph, etc. Our concern in this paper is about connected planar graph.

Connected graph, is a graph that connected, a graph that have no isolated vertex. Isolated vertex is a vertice that have no edge incident with it. But not only isolated vertex, a graph is not a connected graph if it can be divided into two or more subgraphs that are disjoint. So, in a connected graph, we can go from any vertice v_i to any vertice v_j.

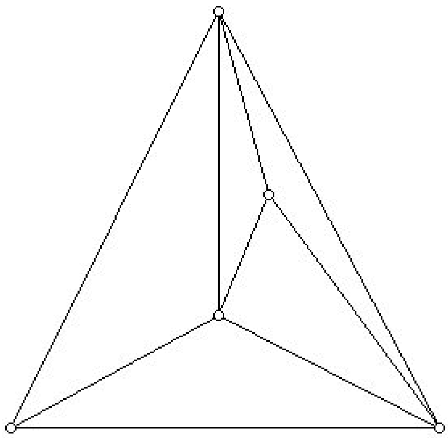

Fig. 3 Example of a planar graph
www.ams.org/samplings

A graph is planar if it can be embedded in the plane, or it can be drawn in a plane in some way such that no two edges are crossing each other. Note that if some graph in its geometric representation still have two crossing edges, it can't be straightly concluded that the graph is not planar, because there are possibilities that if we draw the same graph in different geometric representation, we may get no two crossing edges. A planar graph that is drawn such that there are no two crossing edges is called plane graph.

There is a theorem to check wether any given graph is a planar graph. The theorem called Kuratowski, due to its inventor Kazimierz Kuratowski. The theorem states that any given graph is a planar graph if and only if that graph had no subgraph isomorphic or homeomorphic to K_5 and $K_{3,3}$.

Two graphs is isomorphic if they are two identic graphs with different geometrical representation. Identic means that they have same number of vertex and edges and there exist a bijection each between the set of edges from the first graph and the set of edges from the second graph, also between the set of vertex from the first graph and the set of vertex from the second graph.

There is a special form of graph called tree, that is, a graph that has no cycle. Tree is a connected graph, with special properties that for any two vertex v_i and v_j, there only exist one unique path from v_i to v_j.

IV. Euler's Formula

In mathematics, specifically in algebraic topology, Euler's formula is a formula that relates the number of vertex, the number of edges, and the number of faces of any convex polyhedra. Euler characteristic x is defined as a topological invariants

$$x = V - E + F$$

The Euler characteristic for convex polyhedra always equals 2

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V-E+F$
Tetrahedron		4	6	4	2
Hexahedron or cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

Fig. 4 Table of Euler characteristic for convex polyhedra
storyofmathematics.com

Where $V, E,$ and F respectively denotes the number of vertex, the number of edges, and the number or faces. The value of Euler characteristic is 2 for any convex polyhedra, but varies over any non convex polyhedra or other geometrical structure. Euler characteristic for Klein bottle and torus is 0, for tetrahemihexadron is 1, for cubohemioctahedron is -2, etc.

It is known that every polyhedral has its equivalent planar graph representation. Thus, we can make some more specific formula for any planar graph. Euler's formula for connected planar graph states that any connected planar graph has Euler characteristic 2. There are many proofs had published about this formula. One of the example is Cauchy's proof, using stereographic projection to map a plane into two-dimentional sphere. This proof uses theorems from advanced mathematics. In this paper, we are going to prove Euler's formula in connected planar graph using simpler method, by mathematical induction.

V. Proof of Euler's Formula

As stated before, there had been many approach trying to prove Euler's formula. The results of Euler's formula for connected planar graph are used to prove other result in algebraic topology. But this time, we are just going to prove the theorem for connected planar graph.

Let $G = (V, E)$ is a finite, connected planar graph drawn without any crossing edge. Let V, E, and F respectively denotes the number of vertex, the number of edges, and the number of faces (including the external face). Prove that the Euler characteristic for G is 2, that is

$$V - E + F = 2$$

A. Induction on edges

First, let the number of vertices of the graph V be some constant k, where k is any natural number, $k \geq 1$.

$$V = k, k \in N, k \geq 1$$

Because k is finite and G is a simple graph, then the finite property of G is already fulfilled. Now, because the graph G must be a connected graph, then the minimum number of edges needed is $k - 1$.

$$E \geq k - 1$$

Now, as the first step of mathematical induction, we must first solve for the base case.

Because we use induction on edges and the minimum number of edges is $k-1$, then the base case is $E=k-1$. That is, G is a tree.

1. Base case

If $E=k-1$ and $V=k$, then G is a tree, that is, G has no cycle.

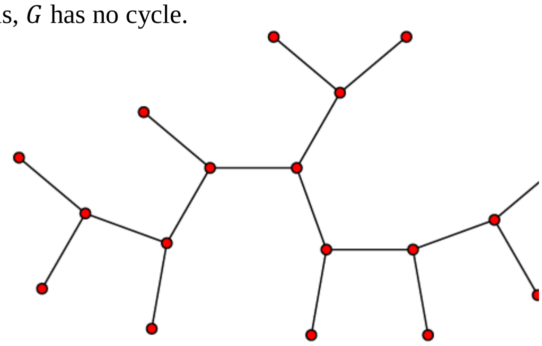

Fig. 5 Graph representation of tree
quora.com

As seen from the picture above, because tree has no cycle, thus the only face exists is the plane itself. Stated in other way

$$F=1$$

Thus,

$$V-E+F=k-(k-1)+1=2$$

That is, we already proved the formula for base case.

2. Inductive case

We are going to use simple mathematical induction in this proof. Because of that, we are only going to assume that for some $n \geq k-1$, $n \in \mathbb{N}$, then $G=(V,E)$ where $V=k$ and $E=n$ is a finite, connected planar graph. Now, we are going to check the formula when $E=n+1$.

Because n is finite, then it is clear that $n+1$ is also finite. Because with n edges, G is a connected graph, then we only need to add 1 edge anywhere and the graph is still connected. Last, if n is a maximum number of edges G can have to form a planar graph, then we must not check the graph of $n+1$ edges, so we can asumme directly that we can add 1 edge somewhere in a graph with n edges to still form a planar graph.

Because the new edge doesn't intersecting any other edge, then the edge must be dividing some face into two spaces. The corollary of this is that the number of faces also adding up by 1. Concludedly, adding one edge to a graph make the number of faces also added up by 1, which leaves the value of $V-E+F$ invariant, as long as the condition is satisfied.

As an addition, we already have some result in planar graph, also according to Leonhard Euler that for any planar graph there exists an inequality

$$E \leq 3V-6$$

So that the number of edges can not march to infinity. And because the value of $V-E+F$ is unchanged, we already proved that for all finite, connected planar graph

$$V-E+F=2$$

B. *Induction on vertex*

Using induction on vertex, we don't have to set neither the number of edges nor the number of faces constant. As usual, we have to solve the base case. In this case, because the induction is on the vertex, then the base case is when the number of vertex is 1, because V can not be an empty set due to the definition of graph itself.

1. Base case

If $V=1$, then it is clear that G is a null graph, that is, a graph with no edge.

$$E=0$$

The graph G is just consists of one vertice, so the faces is just the plane itself. Thus,

$$F=1$$

If we put these numbers into the Euler's formula, then

$$V-E+F=1-0+1=2$$

That is, we already proved the formula for base case.

2. Inductive case

Let V, E, and F is such that the graph G is a finite, connected planar graph. Assume that with $V=k$ vertex, $k \in \mathbb{N}, k \geq 1$, the Euler's formula is satisfied. Similar to the previous proof by induction on edges, we are going to use simple mathematical induction.

For the inductive case, consider a graph G' with $V'=k+1$ vertex. Let the graph G' is constructed by adding one vertice to a graph G with $V=k$ vertex. By assumption, G satisfies the Euler's formula.

Adding one vertice without adding any edge to a graph would make the new graph not a connected graph. Because G' must be a connected grap, then after adding a vertice,we must also adding some edges to make the new graph connected. Because the number of vertex of G is k, the number of new edges added to the graph can not exceed k. Stated in other words, if $e \in \mathbb{N}$ denotes the number of edges added to G and incident to the new vertice. Then

$$1 \leq e \leq k$$

Adding one edge would not change the number of faces. Adding two edges would make a new cycle in the graph, thus make one new face out of the plane. Continuously, adding e edges would make $e-1$ new faces. Thus,

$$E' = E + e$$
$$F' = F + e - 1$$

Inserting these numbers into the Euler's formula, we would get

$$V' - E' + F'$$
$$= V + 1 - (E + e) + F + e - 1 = 2$$

Using previous assumption that $V - E + F = 2$. Once again, we already proved that for all finite, connected planar graph

$$V - E + F = 2$$

VI. Conclusion

Mathematical induction is a powerful tool in proofing technique. It also already being showed that induction can also be applied in graph, not only natural numbers. By looking for the simplest technique, we had already proved the famous Euler's formula for connected planar graph.

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Pernyataan

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Dewita Sonya Tarabunga
13515021