Nash Equilibrium on the Prisoner's Dilemma problem

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***Abstract*—Game theory is one of the applications in discrete mathematics focusing on the decision making process and analysis of strategies for dealing with competitive situation where the outcome of a participant's choice of action depend critically on the action of other participants. The effectiveness of game theory making it has been widely used in another subject beside mathematics and informatics, like business, biology, physics and even in a war. Some of the situation will result in an equilibrium namely Nash equilibrium that will further discussed later along with one of it's simple example, Prisoner's dilemma also with it's applications in real life.**

***Index Terms*—Game theory, Nash equilibrium, Prisoner's dilemma.**

I. INTRODUCTION

Game theory is the science of strategy and decision making. It models a situation in real life whether it's in biology, chemistry, psychology or even economics into a mathematical model of conflict and cooperation between intelligent rational decision-makers. It also discussed the decisions or steps of decision-makers who are aware of the fact that their steps affect the payoffs or results for all, including theirself, therefore influence the decisions of other decision-makers. The games itself range from chess to child rearing and from tennis to takeovers. But, the games all share the common feature of interdependence. That is, the outcome for each participant depends on the choices of all. In so-called zero-sum games the interests of the players conflict totally, so that one person's gain always is another's loss. More typical are games with the potential for either mutual gain (positive sum) or mutual harm (negative sum), as well as some conflict.

Game theory has been used in a wide-range of subjects, from games like poker, tic-tac-toe, or chess to solving serious matters like biological problems of population, predator-prey mechanism and also economic things such as auctions, pricing, competitive business and others. What make this theory so applicable is the flexibility of this theory so it can model a various kind of problems into a set of strategies.

In a more specific matter, there is a solution concept of non-cooperative game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing their own strategy. If each player has chosen a strategy and no player can benefit by changing strategies while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a Nash equilibrium.

Nash equilibrium will be used in a vary field but to easily model and demonstrate it, we will use Prisoner's dilemma, a puzzle of that has been popular among game theorists for its unique nature. It shows the nature of human, why two individuals might not cooperate, even if it appears that it is in their best interest to do so.

II. FUNDAMENTAL THEORIES

A. *Game Theory*

Game theory is the science of strategy. It attempts to determine mathematically and logically the actions that "players" should take to secure the best outcomes for themselves in a wide array of "games". The games it studies range from chess to child rearing and from tennis to takeovers. But the games all share the common feature of interdependence. That is, the outcome for each participant depends on the choices of all. In so-called zero-sum games the interests of the players conflict totally, so that one person's gain always is another loss. More typical are games with the potential for ehtier mutual gain (positive sum) or mutual harm (negative sum), as well as some conflict.

Game theory was pioneered by Princeton mathematician John von Neumann. In the early years the emphasis was on games of pure conflict (zero-sum games). Other games were considered in a cooperative form. That is, the participants were supposed to choose and implement their actions jointly. Recent research has focused on games that are neither zero sum nor purely cooperative. In these games the players choose their action separately, but their links to others involve elements of both competition and cooperation. Game theory was developed extensively in the 1950s by many scholars. Game theory was later explicitly applied to biology in the 1970s, although similar developments go back at least as far as the 1930s. Game theory has been widely recognized as an important tool in many fields. Eight game-theorists have won the Nobel Memorial Prize in Economic Sciences, and John Maynard Smith was awarded the Crafoord Prize for his application of game theory to biology.

Games are fundamentally different from decisions made in a neutral environment. To illustrate the point, think of the difference between the decisions of a lumberjack and those of a general. When the lumberjack decides how to chop wood, he does not expect the wood to fight back; or we could say that the environment is neutral. But when the general tries to attack the enemy's army, one must anticipate and overcome resistance to his plans. Like the general, a game player must recognize his interaction with other intelligent and purposive people. His own choice must allow both for conflict and for possibilities for cooperation.

The core of the game is the interdependence of player strategies. There are two different types of strategic interdependence, namely sequential and simultaneous. In the former the players move in a sequence, each aware of the other's previous actions. In the later, the players act at the same time, each ignorant of the other's actions.

A general theory for a player in a sequential game is to "look ahead" and "reason back". Each player should figure how the enemies will respond to their current move, how they will respond in turn, and so on. The player anticipate where his initial decisions will ultimately lead and uses this information to calculate his current best choice. When thinking about how other will respond, he must think as they would; he should not impose his own reasoning on them.

In principle, sequential games that ends after a finite moves can be solved completely. We determine each player's best strategies by looking ahead to every possible outcome. Simple games, such as rock-paper-scissors can be solved this way and therefore are not challenging. For many other bigger games like chess, the calculations are too complex to be calculated even for computers to do so. Therefore, the players look a few moves ahead and try to evaluate the resulting positions on the basis of experience.

In contract to the linear chain of reasoning for sequential games, a game with simultaneous moves involves a logical circle. Although the players act at the same time, in ignorance of other current actions, each must be aware that there are other players who are similarly aware, and so on. Therefore, each must figuratively put himself in the shoes of all and try to calculate the outcome. His own best action is an important part of this overall calculation.

B. Nash Equilibrium

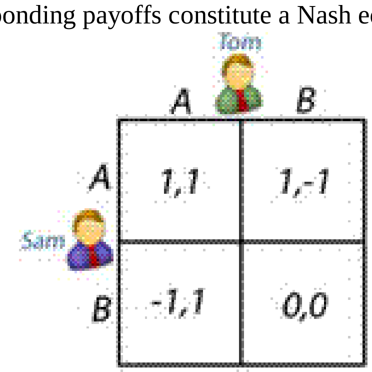
The theory construct a notion of "equilibrium" to which the complex chain of thinking about thinking could converge. Then the strategies of all players would be mutually consistent in the sense that each would be choosing his or her best response to the choices of the others. Nash used novel mathematical techniques to prove the existence of equilibrium in a vary general class of games.

Nash equilibrium is a fundamental concept in the theory of games and the most widely used method of predicting the outcome of a strategic interaction in the social sciences. A game consists of the following three elements : a set of players, a set of action available to each player and a payoff function for each player. The payoff functions represent each player's preferences over action profiles, where an action profile is simply a list of actions, one for each player. A pure strategy Nash equilibrium is an action profile with the property that no single player can obtain a higher payoff by deviating unilaterally from this profile.

The Nash equilibrium was named after John Forbes Nash, Jr. A version of the Nash equilibrium concept was first known to be used in 1838 by Antoine Augustin Cournot in his theory of oligopoly. In Cournot's theory firms choose how much output to produce to maximize their own profit. However, the best output for one firm depends on the output of others. A cournot equilibrium occurs when each firm's output maximizes its profits given the output of the other firms, which is a pure strategy Nash equilibrium. Cournot also introduced the concept of best response dynamics in his analysis of the stability of equilibrium.

The modern game-theoretic concept of Nash equilibrium is instead defined in terms of mixed strategies, where players choose a probability distribution over possible actions. The concept of the mixed strategy Nash equilibrium was introduced by John von Neumann and Oskar Morgenstern in their 1944 book The Theory of Games and Economic Behavior. However, their analysis was restricted to the special case of zero-sum games. They showed that a mixed strategy of Nash equilibrium will exist for any zero-sum game with a finite set of actions. The contribution of John Forbes Nash in his 1951 article Non-cooperative Games was to define a mixed strategy Nash equilibrium for any game with a finite set of actions and prove that at least one Nash equilibrium is exist in such a game.

In game theory, the Nash equilibrium is a solution concept of a non-cooperative game involving two or more players, in which each player is assumed to know the equilibrium strategies of other, and no player has anything to gain by changing only their own strategy. If each player has chosen a strategy and no player can gain benefit by changing strategies while the other players keep their unchanged, then the current set of strategies and the corresponding payoffs constitute a Nash equilibrium.



Picture 2.1 Example of payoff matrix referring to Nash equilibrium (http://investopedia.com)

In a simple example, Joshua and Teo will be in a Nash equilibrium if Joshua is making the best decision he can, taking into account Teo's decision and Teo is making the best decision he can, taking into account Joshua's decision. Likewise, a group of players are in Nash equilibrium if each one is making the best decision that he or she can, taking into account the decision of the others.

C. Prisoner's Dilemma

The prisoner's dilemma is an example of a game analyzed in game theory that shows why two individual might not cooperate, even if it appears that it is in their best interests to do so. It was originally framed by Merril Flood and Melvin Dresher working at RAND in 1950. Albert W. Tucker formalized the game with prison sentence rewards and naming it, presenting it as follows:

> "Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of speaking to or exchanging messages with the other. The police admit that they don't enough evidence to convict the pair on the principal charge. They plan to sentence both to two year in prison on a lesser charge. Simultaneously, the police offer each prisoner a bargain. Each prisoner is give the opportunity either to betray the other, by testifying that the other committed the crime, or to cooperate with the other by remaining silent"

The offer which police offer are :

1. If A and B both betray the other, each of them serves 2 years in prison.
2. If A betrays but B remain silent, A will be free and B will serve 3 years in prison. If A remain silent but B betrays, A will serve 3 years in prison and B will be free.
3. If a and B both remain silent, both of them will only serve 1 year in prison.

It's implied that the prisoners will have no opportunity to reward or punish their partner other than the prison sentences they get, and that their decision won't affect their reputation in future. Because betraying a partner offers a greater reward than cooperating with them, all purely rational self-interested prisoners would betray the other, and so the only possible outcome for two purely rational prisoners is to betray each other. The interesting part of this result is that pursing individual reward logically leads both of the prisoners to betray, when they would get a better reward if they both cooperated. In reality, humans display a systematic bias towards cooperative behavior in this and similar games, much more so than predicted by a simple models of rational self interested action. A model based on a different kind of rationality, where people forecast how the game would be played if they formed coalitions and then they maximize their forecasts, has been shown to make better predictions of the rate of cooperation in this and similar games given only the payoffs of the game.

The prisoner's dilemma game can be used as a model for many real world situations involving cooperative behavior. In casual usage, the label "prisoner's dilemma" may be applied to situations not strictly matching the formal criteria of the classic or iterative games : for instance, those in which two entities could gain important benefits from cooperating or suffer from the failure to do so, but find it merely difficult or expensive, not necessarily impossible, to coordinate their activities to achieve cooperation.

D. Tree and Decision Tree

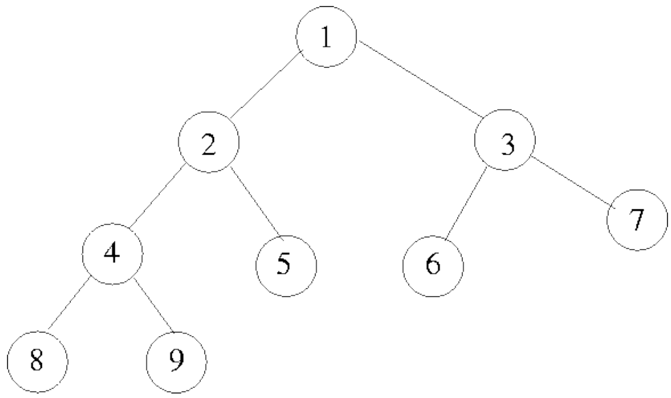
In discrete mathematics, specifically in graph theory, a tree is an undirected graph in which any two vertices are connected by exactly one simple path. In other words, any connected graph without simple cycles is a tree. While a forest is a disjoint union of trees.

The various kinds of data structures referred to as trees in computer sciences are equivalent as undirected graphs to trees in graph theory, although such data structure are generally rooted trees, thus in fact being directed graphs, and may also have additional ordering of branches.

A tree is an undirected simple graph G that satisfies any of the following equivalent conditions :

1. G is connected and has no cycles.
2. G has no cycles and a simple cycle is formed if any edge is added to G.
3. G is connected, but is not connected if any single edge is removed from G.
4. G is connected and the 3-vertex complete graph K3 is not a minor of G.
5. Any two vertices in G can be connected by a unique simple path.

Another terms we should know is n-ary tree. An n-ary tree is a rooted tree for which each vertex has at most n children. 2-ary trees are sometimes called binary trees.

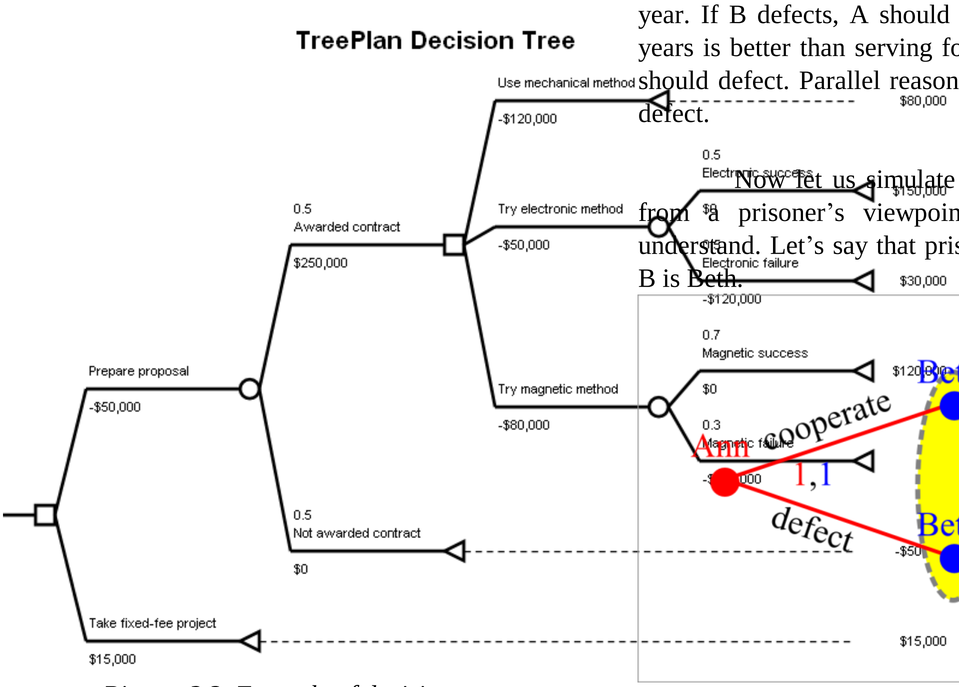


Picture 2.2 Example of binary tree (http://lcm.csa.iisc.ernet.in)

The development of the tree data structure that we will use is a decision tree. A decision tree is a decision support tool that uses a tree-like graph or model of decisions and their possible consequences, including chance event outcomes, resource costs, and utility. It is one way to display an algorithm.

Decision trees are commonly used in operations research, specifically in decision analysis, to help identify a strategy most likely to reach a goal.

A decision tree is a flowchart-like structure in which internal node represent test on an attribute, each branch represents outcome of test and each leaf node represents class label. A path from root to leaf represents classification rules.

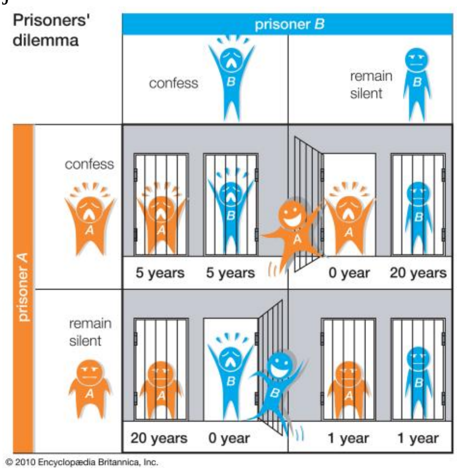


Picture 2.3 Example of decision tree (http://treeplan.com)

III. DISCUSSION

Let us take a deeper look on the Prisoner's dilemma case that has been stated at chapter 2. To get a bigger picture of this case, it is easier for us to make this into a matrix called *payoff matrix*. Payoff matrix is a term referring to a decision analysis tool that summarizes pros and cons of a decision in a tabular form. This tool lists all payoffs with all possible combinations of alternative actions and external conditions.

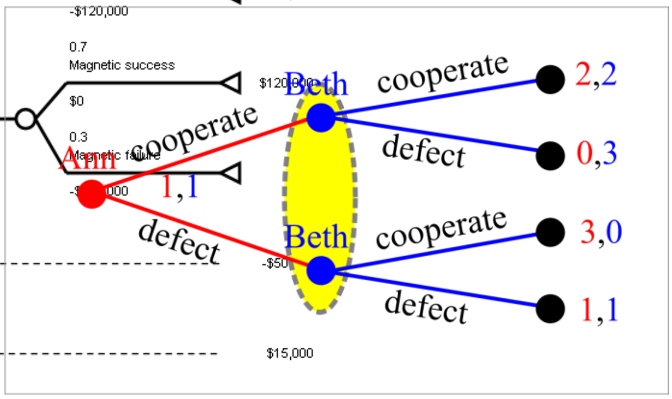
A payoff matrix referring to Prisoner's dilemma case is just like below :



Picture 3.1 The payoff matrix of Prisoner's dilemma (http://beranger.org)

Here, regardless of what the other decides, each prisoner gets a higher pay-off by betraying the other (defecting). The reason involves an argument by dilemma : B will either cooperate or defect. If B cooperates, A should defect, since going free is better than serving 1 year. If B defects, A should also defect, since serving 1 years is better than serving for 5 years. So either way, A should defect. Parallel reasoning will show that B should defect.

Now let us simulate this case in a decision tree from a prisoner's viewpoint so it can be easier to understand. Let's say that prisoner A is Ann and prisoner B is Beth.



Picture 3.2 Decision tree of Prisoner's dilemma (http://eprisner.de)

From a decision tree above, we can get a clear point of view that whatever Ann choose, it will be better for Beth to defect (3>0, 1>0). This is also applies to Ann if we use her point of view.

So what is the relation between Prisoner's dilemma and Nash equilibrium? We can see from the situation that both prisoner improve their own situation by switching from cooperating to defecting, given knowledge that the other prisoner's best decision is to defect. Thus, the Prisoner's dilemma has a single Nash equilibrium, that is both players choosing to defect.

What has long made this an interesting case to study is the fact that this scenario is globally inferior to "both cooperating". That is, both players would be better off if they both choose to "cooperate" instead of both choosing to defect. However, each player could improve his own situation by breaking the mutual cooperation, no matter how the other player possibly changes their decision.

There are many real life examples of Prisoner's dilemma. Take a look at the case of doping in sport. If two competing athletes have the option to use an illegal and dangerous drug to boost their performance, then they must also consider the likely behavior of their competitor. If neither athlete takes the drug, then neither gains an advantage. If only one does, then that athlete gains a significant advantage over their competitor. If both athletes takes, however, the benefits cancels out and only the drawbacks remain, putting them both in a worse position. We could also look at the Cold war. During the cold war, the opposing alliances of NATO and Warsaw Pact both had the choice to arm or disarm. From each side's point of view, disarming whilst their opponent continued to arm would have led to military inferiority and possible annihilation. If both sides chose to arm, neither could afford to attack each other, but at the high cost of maintaining. If both sides chose to disarm, war would be avoided and there would be no costs.

IV. Conclusion

Game theory can be widely used in many subject and fields like philosophy, economics, social and others. One of the most important theorem in game theory is Nash equilibrium which is solution concept of a non-cooperative game involving two or more players, in which each player is assumed to know the equilibrium strategies of other, and no player has anything to gain by changing only their own strategy. Prisoner's dilemma is an easy and simple example to demonstrate this Nash equilibrium theory and also it's application in real world and everyday life.

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Bandung, December 17th 2013

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