# Graph and Logic Theorem within Game Theory

## Focusing on Braess's Paradox and Prisonner's Dilemma Puzzle

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***Abstract*—Game theory has already been long known as the science of strategic decision making. As one of many topics in the field of discrete structure as well as economics, game theory is widely-used in solving many games as well as problems arise in real world events. Some of these problems are solved using the help of graph or logic theorem alongside the game theory itself. This paper will specially discuss problems which meet this criteria, namely the use of game theory along with the graph or logic theorem. Two famous problems that will be further discussed in the paper are known as Braess's Paradox and Prisonner's Dilemma.**

***Index Terms*—Game theory, graph theorem, logic theorem, Braess's Paradox, Prisonner's Dilemma**

## I. Introduction

Game theory is a study of concepts used in strategic decision making for the situations of cooperation and non-cooperation under some specific rules[1]. It discusses the decisions or steps of decision-makers who are aware of the fact that their steps affect the payoffs/results for all, including himself, therefore influence the decisions of other decision-makers[3]. A situation wouldn't be considered an appropriate modeling by game theory if the strategies or decisions of one particular side is insignificant or doesn't affect the decisions of others. A good example for further understanding of game theory is the Cold War between the West (US) and the East (Uni Sovyet). In this example, both sides are faced with the decision of arming or disarming its nuclear warfare. The US knows that the Soviet's decision is taking into account the forecast/prediction of the US decision. Meanwhile, the combination of the decision of both sides (arm-arm, arm-disarm, disarm-arm, disarm-disarm) results in different payoffs for the world, including themselves. However, the US is not in a game with Marocco, because the actions of Marocco is insignificant to the US's actions, and Marocco decide its actions regardless its impact on the US policies (note that even the impact wouldn't be significant either, at least in the matter of nuclear warfare).

Game theory is used in wide-range of problems, ranging from games such as tic-tac-toe, chess, to solving biological problems such as migration and predator-prey mechanism, as well as economic problems such as pricing and auctions. It has so much flexibility that most problems involving the interactions of more than one participant can be related as one of it's application.

However, there are problems that discussed in such way that it involves other branch of discrete structure. The Cold War example. as it can be predicted, involves the logic theorem, which will be further discussed later in this paper. Meanwhile, another example, the game tree of tic-tac-toe, is taking into account application of tree (hence, the graph theorem as well) in its explanation. There are various other theorems in the field of discrete structure that intersect the game theory in problems solving or discussion, but it is beyond the scope of this paper.

The Prisonner's Dilemma has been a famous problem/puzzle between game theorists as well as the public for its unique nature. Focusing on the decision of betrayal and cooperates, it brings a question of the nature of human being. Would us cooperate for a chance better result? Or would us choose betrayal for a guaranteed safe choice? Those are questions that will be answered and further explained in a section later in this paper. Furthermore, as a basic concept that has been around since 1950 the Prisoner's Dilemma also provide early development of other real life problems such as the Cold War example mentioned above as well global warming problems (environmental studies) and doping (sport).

While the Prisonner's Dilemma puzzle is more of a concept that can be further generalized into different application in real life problems, the Braess's Paradox is brought up from real life problem and hence, straightly solved for a better solution on the real life application. Mostly touch the field of transportation science, the Braess's Paradox has become the base of policies regarding traffic flow planning as well as street addition/reduction in several cities around the world. This too, is a famous problem that author will cover in this paper.

## II. TERM AND DEFINITION

This section will covers the key terms and definitions used in this paper, mainly to avoid misunderstanding that would happen if clear explanations of terms and definitions used are unprovided.

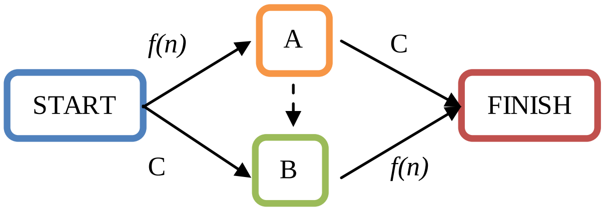
First of all, as the first section explain, game theory is a study of concepts used in strategic decision making for the situations of cooperation and non-cooperation under some specific rules. Game itself can be defined as an activity among two or more independent decision-makers seeking to achieve their objectives in some limiting context[1], while theory (as in science) is a set of thinking that provide rational explanation based on evidence[5]. Game often contrived as a form of entertainment, but as in this paper , the term game as in game theory will not be strictly related to entertainment or enjoyment alone.

One of games analyzed in game theory is the infamous Prisonner's Dilemma puzzle. It is a puzzles that shows why two individuals might not cooperate, even if it actually benefits both party. A closest sample to a Prisoner's Dilemma problem would be :

> Dean and Jack have been arrested for the robbing of a local Bank. In the police office, they are placed in separate isolation cells. As a robber, both doesn't care much about other and put their personal advantage above everything. Knowing this, a clever prosecutor give both criminals a same offer. Each may choose to confess or remain silent. If one confess and the other remains silent than the one who confess will be free of charges, and the silent one will be convicted and surely get a serious punishment. If one silent and the other confess than it will go likewise (but inversely). If both of them confess than he will give them early parole as reward, and if both remain silent, he'll have to charge them only for firearms possession charges, which will be the lightest punishment of all. It must be remembered that they are separated in different isolation cells, making it's impossible to discuss their decision[6].

The so-called "dilemma" faced by the prisoners here is that, they have to choose between a self-interest action or giving a shot for cooperative action to reach better result. Further explanation will be discussed by using logic theorem, in later section. On the history part, though, this problem was devised by Merrill Flood and Melvin Dresher in 1950. The title "prisoner's dilemma" and the theme of prison are invented by Albert Tucker, who strive in making the puzzle more accessible to Stanford psychologists. In the sixties and seventies this problem is highly popular, based on the number of paper published related to it from variety of disciplines, and has never shown any signs of abating [6].

Braess's Paradox, however, is an entirely different problem/puzzle. Related to transportation science as well as flow network, Braess's Paradox concentrating on the unique paradox that in some certain cases, adding a new road in an already congested network wouldn't make the condition better but surprisingly even slow the already congested traffic even more. The following diagram shows one of simple Braess's Paradox example.




*Picture 2.1 Diagram of a simple Braess's Paradox*

> Suppose there are *n* drivers who wish to travel fro START to FINISH and therefore has to choose between route START-A-FINISH and START-B-FINISH. The travel time from START to A and B to FINISH are equal, determined by a function *f(n)* < C based on the number of traffic (cars) that choose to drive onto the road. The travel time from START to B and A to FINISH are a equal and constant, can be seen as a wide highway which doesn't affected by the amount of traffic on it. To reduce the congested traffic in both route, the City Council decided to make a shortcut from A to B which takes negligible time (or even you can say 0 time) to travel. After the try, surprisingly it makes the travel time even slower or in other words make the congested road even worse. What was wrong? Why adding a shortcut to the road makes the traffic even worse?

An alternate viewing of the will goes by seeing that the additional link is present, and while the traffic has reached its equilibrium, there exists some other distribution of flows for which some travelers decrease their travel time but no travelers increased their travel time than what they have in the equilibrium. [7]

## III. RELATED THEORIES

### A. Logic Theorem

Logic theorem is the base for all reasoning. It is related to the relation of statements, and has been widely used in many other disciplines as well as real-life application [1].This subsection covers some parts of logic theorem that will be used mainly in this paper, namely proposition, logic law, implication and bi-implication, as well as arguments.

### A.1. Proposition

Proposition is an expression in language or signs of something that can be either true or false[1]. A proposition value is exact, either true or false, and cannot be both. Propositions are usually labeled with lower case letter such as

p : My pencil is black,.

q : Indonesia is a republic.

r : 3+5 = 8.

There are times that there are no connections whatsoever between two or more proposition in a valid sentence. For example, a weird sentence such as "If my pencil is black, then Indonesia is a republic," is a proposition, regardless its value.

### A.2. Logic Law

Logic law, also known as the law of proposition algebra is a series of law that covers the equivalent logic relationship of propositions. Some may looks insignificant, but all of these statements are actually very important, ultimately in building arguments, implications, and conclusions. Below are some of the logic laws.

Identity Law : p OR F = p ; p AND T = p
Null Law : p AND F = F; p OR T = T
Negation Law : p OR ~p = T ; p AND ~p = F
Idempotent Law : p OR p = p ; p AND p = p
Involution Law : ~(~p) = p
Absorption Law : p OR (p AND q) = p
p AND (p OR q) = p

Commutative Law :
p OR q = q OR p
p AND q = q AND p

Associative Law:
p OR (q OR r) = (p OR q) OR r
p AND (q AND r) = (p AND q) AND r

Distributive Law :
p OR (q AND r) = (p OR q) AND (p OR r)
p AND (q OR r) = (p AND q) OR (p AND r)

De Morgan's Law :
~(p AND q) = ~p OR ~q
~(p OR q) = ~p AND ~q

p and q and r are premises (propositions), ~p are the negation of p, and either OR or AND is a boolean operator.

### A.3. Implicatios and Bi-Implication

The concept of logical implication involves a specific logical function, a specific logical relation, and the various symbols that are used to denote this function and this relation[7]. The form of implication in its simplest form is If (conditions) then (consequences). Meanwhile there are other means of linguistic representation of implication such as p *implies* q, and some others which will not be discussed further in this paper. In the form ``if p then q '', the first term, p , is called the antecedent and the second term, q , is called the consequent. The whole statement itself is called the conditional. If we assume that the conditional statement is true, we can safely say that the antecedent is a sufficient condition for the consequent, while the consequent is a necessary condition for antecedent, all in truth value. We will use logical implication as our tools to discuss the puzzles mentioned earlier. The truth table for implication is :

| p | q | p—>q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

*Table 3.1 Implication's Truth Table*

While there is implication, in logic theorem there isa also a term known as bi-implication. Usual interpretation of it is by the means of linguistic "…if and only if… ". Thus, biimplication in its simplest form will be "p if and only q".

The truth table for bi-implication is:

| p | q | P<—>q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

*Table 3.2Biimplication's Truth Table*

which clearly shows that biimplication value holds true if p and q share a same value either it is false or true.

### A.3. Arguments

Argument refers to a list of proposition that consist of hypotheses/premises and the conclusion. An argument can be either valid or invalid. In this lecture, arguments is valid if the conclusion will hold true for all of true hypotheses, and will be false otherwise. Below are some arguments that are proven valid :

| Modus Ponen | | Modus Tollen |
|---|---|---|
| p → q | | p → q |
| p | | ~q |
| ---------- | | ---------- |
| q | conclusion | ~p |

| Syllogism | | Disjunctive Syllogism |
|---|---|---|
| p → q | | p OR q |
| q → r | | ~p |
| ---------- | | ---------- |
| p → r | conclusion | q |

### *B. Graph Theorem*

Graph is used to represent discrete objects and the relations of them. Visual representation of graph can be reached by stating object as a node and the relationships between them with a line/edge[1]. Mathematically, graph is defined by :

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DEFINITIONS 3.1.

Graph G is defined by a pair set (V,E), which in this case :

V = a non empty set of vertices or nodes = (v1,v2,v3,…)

and

E = a set of edges or arcs which connect a pair of nodes = (e1,e2,e3,…)

or shortly written as G = (V,E)

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Graph can be grouped to be several categories, depending on the point of view. In this paper, we will use several categories of graphs, which is

a. *Directed graph* or *digraph*

A type of graph so that each of its sides is given an orientation direction is called a *directed graph* or *digraph.* An oriented edges/sides is usually known as *arcs*. In *directed graph*, $(v_j,v_k) \neq (v_k,v_j)$. In $(v_k,v_j)$, $v_k$ is called *initial vertex,* while $v_j$ is called *terminal vertex.*

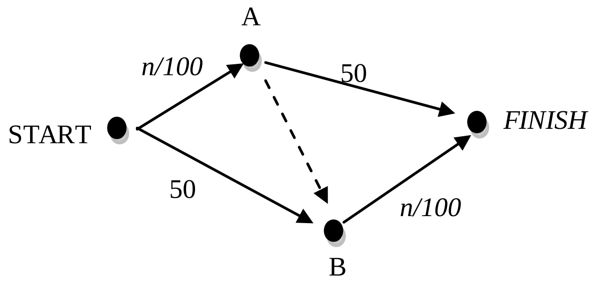
b. *Weigthed Graph*

A graph that each sides is given a value is called a *Weighted Graph.* The value in each sides/edges can be a distance, travel cost, or in this paper will be the travel time between to place (nodes).

## IV. ANALYSIS AND DISCUSSION

### *A. The Braess's Paradox*

Recall the Braess's Paradox mentioned in section 2. The *n* will be 5000, meaning there are 4000 travelers who wish to go from START to FINISH. The travel time between START to A or B to FINISH is a function f (n), which will be the number of load traffic (n) divided by 100. The constant C will be 50, meaning it needs 50 minutes to reach START to B or A to FINISH. With *Directed Graph* and *Weighted Graph,* we obtain a graph representing the problem just like the following picture :




*Picture 4.1 Diagram of the Braess's Paradox*

Let's say that the shortcut A-B doesn't exist initially. Thus, the P drivers who took START-A-FINISH route (P is the amount of drivers who choose to took this route) will have the travel time of

**(P/100) + 55**

compared to

**(Q/100) + 55**

of Q drivers who took START-B-FINISH.

It is important that either route is not shorter/longer, as if so, it would be an equilibrium, as a rational driver would switch from longer to shorter route.

Now, as there is 5000 drivers, the fact that A+B =5000 implies that the equilibrium reached when A=B=2500. This again, implies that each route takes

**(2500/100) + 55 = 80 minutes**

After this, suppose that the shortcut has been built, and it's so negligible that it can be safely said that it takes 0 minutes to travel through the shortcut. All drivers will then choose the START-A route rather than the START-B route, since Start-A will only take

**5000/100=50 minutes**

at its worst, whereas driving in START-B is fixed to take **55 minutes**. When they reach A, every rational driver will switch to take the shortcut A-B because it seems somewhat faster, and from there taking B-FINISH route, which at worst take

**5000/100=50 minutes**

while A-FINISH take fixed time of 55 minutes. Now the supposedly 'faster' travel time of each driver is

**50+50 =100 minutes,**

which is 20 minutes longer than the initial travel time!

In this condition though, no driver want to switch route, as the original routes (START-A-FINISH and START-B-FINISH) is now take at most

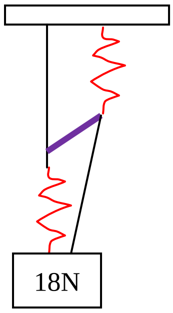
**(5000/100)+55=105 minutes**

But then, if every driver agreed not to use the shortcut A-B, or the shortcut is closed, the equilibrium will be reached again, and they would be benefit by reducing their travel time by 20 minutes. However, because any single driver (only) will always benefit by taking the shortcut, the socially optimal distribution is not stable and thus, the Braess's paradox occurs.

Example above is an example of simplest Braess's paradox problem. Another closest variations would be adding a variable length to the edges of the graphs as well as the capacity of each edges. This will later form a flow network or graphs, which far beyond the scope of this paper. Another complex variation is an addition of variable demand from START to FINISH, as well as multiple origin destination pairs, which all can be further seen in the website that the author used for reference.[9][9]

Other application of Braess's Paradox is found in a problem of strings and springs shown below.

Consider the picture below, if all the strings are taut, then each of them will be carrying one-third of the weight of the block. Would the block drop down a bit if the purple string is cut?




*Picture 4.2 The String and Springs Problem*

Most people will answer yes, as it can be deducted easily that if the purple string is cut, than the remaining string will carry more tension and therefore also affect the correspondence spring, resulting in more contracting and thus, a drop on the block. But again, Braess's Paradox states differently.

Let's rethink it all over again. At the initial condition, each string holds 18/3 = 6 N tension and therefore, each spring holds 2*6=12 N tension. When the purple string is cut, each string holds 18/2 = 9 N, and so does the correspondence spring! It means the spring holds less tension (9 compared to 12), and hence, they will contract accordingly, making the block rise[6].

Now, without even trying to, we've already used a lot of logical reasoning in solving these Braess's Paradoxes, for example when we states if the shortcut is open, then all driver will choose the START-A-FINISH routes.

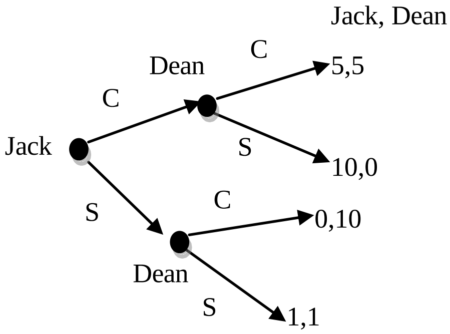
## B. *The Prisoner's Dilemma*

Consider the prisoner's Dilemma problem in section 2. From the offer of the prosecutor, there are four kinds of situations that might arise. Using implications and logical reasoning, if Dean confesses and Jack remains silent, than Dean will be free of charges and Jack will spend considerable amount of time in prison. Meanwhile if Dean remains silent and Jack confesses the inverse situation will happen. However, if both confess, they will each get lighter punishment then they should for their cooperation, although it's still a lot to spend in the jail. The best option will be for both remain silent, as they will be only charged for firearms possessions, being the lightest punishment for best interest of both. If we say free of charges=10, lighter punishment=5, firearms posessions = 1, and max punishments = 0, the following matrices shows the situations.

| Dean\Jack | Confess, | Silent |
|---|---|---|
| Confess | 5,5 | 10,0 |
| Silent | 0,10 | 1,1 |

*Table 4.1 The payoff of Prisoner's Dilemma problem*

The matrices clearly shows that it is actually cooperation (silent, silent) that will benefits the most both of them. The problem is, whatever the other does, each is better off confessing than remaining silent[6]. Say we see the point of view of Dean. If Jack remains silent, than if he remains silent, he would get only charged by firearms posessions, same thing with Jack. But if he choose to confess (betray Jack) then he would be free, neglecting the fact that Jack will spend a lot of time in Jail. Suppose than Jack choose to confess, then Dean clearly wouldn't remain silent as he would be the one to spent his times in the jail. So he would then choose to confess as well. This way of thinking can be shown by a *directed graph* shown below :




*Picture 4.3 Graph of Prisoner's Dilema*

Notice that the graph in Picture 4.3 is actually a tree, more specifically a *decision tree*. Now it is clear that it is best for Dean to choose confess as it's always give better result for him (5>0, 10>1). This also applies to Jack, if we were to see from his point of view.

While this graph (or tree) perfectly simulates the situations arise, it's actually more appropriate to use this kind of form, known as *extensive form* in a sequential games/puzzles such as tic-tac-toe, chess, etc. The Prisoner's Dilemma itself is a simultaneous game, meaning each player decide its steps unknowingly of other player steps. We build the graph above to simulate the way of each prisoner thinking, thus, deciding the best strategy they could pick for the better result of them individually.

It is an interesting way to see that two "rational" prisoners will confess and receive a payoff of 5, while two "irrational" prisoners can cooperate (remains silent) and receive greater payoff of 1[6].

The Prisoner's Dilemma puzzles are often adopted in the economics field. Some of them are a situations when two cigarette company decide to advertise or not, two seller decide to cur price or not, two factory decide to increase production or not, and many others. All of them are applied in similar fashion, and further explanation wouldn't be necessary as it will just be a repetition.

Notice that we have used an enormous amount of logical reasoning and conclusion making to reach this state. Without explicitly said, we have already used the logic theorem thoroughly in this paper.

## V. CONCLUSION

Logic theorem, while not explicitly said, has been the base of all reasoning and explanation. In the the Game Theory, which is widely used in various field of discipline, such as transportation science and economics, logic theorem along with graph theorem play important roles in solving problems such as the Braess's Paradox and the Prisoner's Dilemma. The Braess's Paradox focus in using the *Directed* and *Weighted Graphs*, as well as a big use of logical reasoning, while the Prisoner's Dilemma requires a lot of logical reasoning which includes a lot of conditional propositions (one implies another) , as well as the use of graph in the form of tree.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