Precise Characterizations of the Prisoner's Dilemma Using Mathematical Logic Theory

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***Abstract*— The prisoner's dilemma is a game model for analyzing about decision making. In the prisoner's dilemma problem, the prisoner has to decide whether the he has to keep silent (cooperate with the partner(s)) or to betray the partner(s) considering what decision will his partner(s) make, whether his partner(s) will cooperate with him, or even betray him, so this prisoner can lessening his time in jail. This paper will discuss about some kind of the prisoner's dilemma problems, and the usage of mathematical logic theory in making precise characterizations of the prisoner's dilemma.**

***Keywords*—the prisoner's dilemma, mathematical logic theory, precise characterizations.**

I. Introduction

Logic is the foundation for all reasoning that is based to the relation between some statements. Mathematical logic itself is a subfield of mathematics which include the study of the deductive power of formal proof systems. The mathematical logic theory is often be used in our daily life especially to make a deduction from all statements we have considering about the validity of all statements we have. This theory is usually used by a company to make decision when the company recruiting employees or should they collaborate with the other company. This theory is also used by judges to decide whether the suspected person is really guilty or not.

The prisoner's dilemma is a game model for analyzing about decision making. A classic example of the prisoner's dilemma is presented as two men are arrested, but the police do not possess enough information for a conviction. Following the separation of the two men, the police offer both a similar deal—if one testifies against his partner (betrays/defects), and the other remains silent (cooperates/assists), the betrayer goes free and the cooperator receives the full one-year sentence. If both remain silent, both are sentenced to only one month in jail for a minor charge. If each 'rats out' the other, each receives a three-month sentence. Each prisoner must choose either to betray or remain silent so that he can lessening his time in jail.

In this paper, we will use the mathematical logic theory to make deduction in order to make the best decision in the prisoner's dilemma problem so that decision will give the best result for the prisoner so that he can lessen his time in jail.

II. The Prisoner's Dilemma

2.1 Symmetric 2×2 Prisoner's Dilemma With Ordinal Payoffs

In its simplest form, the prisoner's dilemma described by this payoff matrix

	Cooperate	**Betray**
Cooperate	R,R	S,T
Betray	T,S	P,P

R is the "reward" payoff that each player receives if both cooperate. *P* is the "punishment" that each receives if both betray. *T* is the "temptation" that each receives if he alone betrays and *S* is the "sucker" payoff that he receives if he alone cooperates. We assume here that the game is symmetric, the reward, punishment, temptation or sucker payoff is the same for each player.

2.2 Asymmetry 2×2 Prisoner's Dilemma With Ordinal Payoffs

This type is similar with the symmetric prisoner's dilemma, the difference is we didn't assume that the reward, punishment, temptation, or sucker payoff is the same for each player. The prisoner's dilemma described by this payoff matrix

	Cooperate	**Betray**
Cooperate	Rr,Rc	Sr,Ts
Betray	Tr,Ss	Pr,Ps

2.3 Multiple Moves Prisoner's Dilemma

In general, the prisoner's dilemma is a game in which a “cooperative” outcome only can be gotten when each player decide to neglect their self-interest. Each player only can choose either to cooperate or to betray his partner. In this type, the player can choose not only to cooperate, betray, but also not choosing anything both cooperate and betray.

This prisoner's dilemma described by this payoff matrix

	Cooperate	**Betray**	**Neither**
Cooperate	R,R	S,T	T,S
Betray	T,S	P,P	R,S
Neither	S,T	S,R	S,S

2.4 Multiple Players Prisoner's Dilemma

Most of prisoner's dilemma problem only illustrated by two-player. This type enlarge the structure of the game to a larger groups, changing from two-player to the many-player game. The most obvious generalization for this type is R happens if all cooperate, P happens of all betray, and if some cooperate and some betray, it would pay S for the cooperators and T for the betrayers.

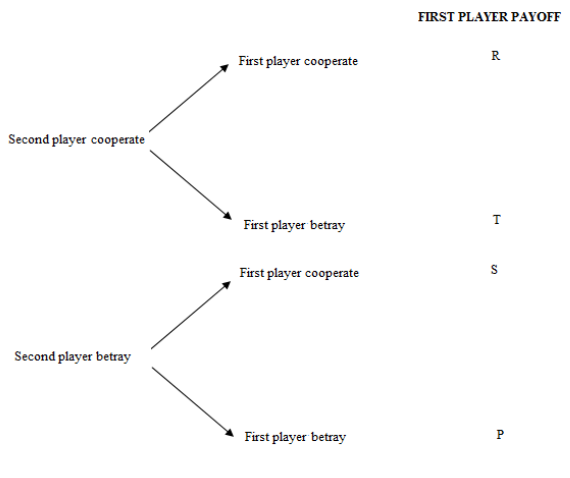
III. Discussion and Analysis

2.1 Symmetric 2×2 Prisoner's Dilemma With Ordinal Payoffs

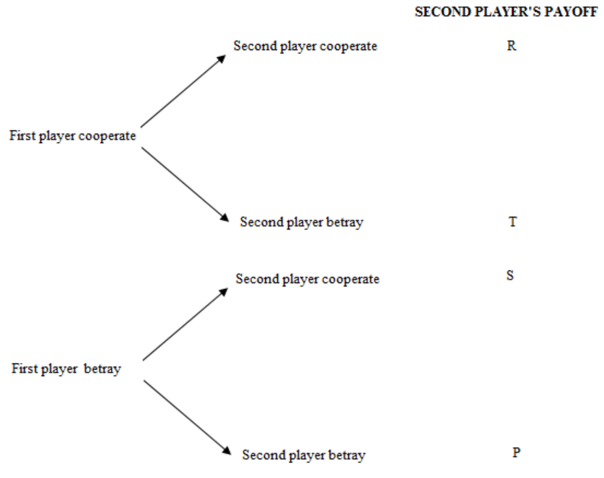
This prisoner's dilemma problem described by this payoff matrix

	Cooperate	**Betray**
Cooperate	R,R	S,T
Betray	T,S	P,P

From the payoff matrix, the first player’s payoff tree diagram can be shown as



The second player’s payoff tree diagram can be shown as



The condition of the payoff is T>R>P>S. There are two players, each has two possible moves, either to ”cooperate” or “betray”. The payoff for each possible moves is shown by the payoff matrix and the payoff tree diagram,.

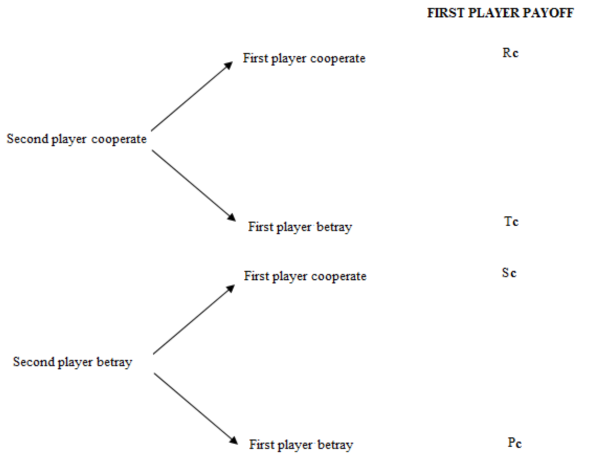
According to the payoff matrix and the payoff tree diagram, if we make assumption that the first player will decide to take the cooperate move, then the second player will get R if he decides to take the cooperate move and get T if he decides to take the betray move, so is better off to take the betray move. If if we make assumption that the first player will decide to take the betray move, then the second player will get S if he decides to take the cooperate move and get P if he decides to take the betray move, and so is again better off taking the betray move. Because of the similarity of the reward, punishment, temptation or sucker payoff for each player, **betray** is also a better choice than cooperate for the second player. Thus, as in a standard treatment in common game theory, if we assume that each player knows how the other values the outcomes and play maximally, those two players will decide to take the betray move and receive a payoff of *P*, than to take the cooperate move and receive greater payoff R.

2.2 Asymmetry 2×2 Prisoner's Dilemma With Ordinal Payoffs

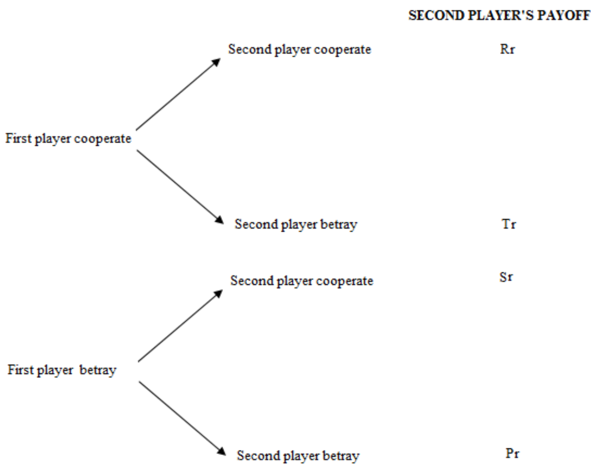
This prisoner's dilemma problem described by the following payoff matrix

	Cooperate	**Betray**
Cooperate	Rr,Rc	Sr,Tc
Betray	Tr,Sc	Pr,Pc

From the payoff matrix, the first player's payoff tree diagram can be shown as



The second player's payoff tree diagram can be shown as



According to the asymmetry prisoner's dilemma problem's payoff matrix, if we assume that the payoffs are ordered as before for each player, that $T_i>R_i>P_i>S_i$ when i= r, c, same as before, players will tend to take the **betray** move than the **cooperate** move considering their self-interest.

Considering the following conditions

a. Tr > Rr and Pr > Sr

b. Tc > Rc and Pc > Sc

c. Rr > Pr and Rc > Pc

If we one of a or b condition happen (not both of them), if we assume that the first player decides to take the cooperate moves, then the second player will get Rr if he decided to take the cooperate move and get Tr if he decided to take the betray move. In the other hand, if we assume that the first player decided to take the betray move, the second player will get Sr for cooperating and Pr for betraying. According to the payoffs the second player will get , it's hard to make a decision whether he has to betray or cooperate. The problem is when he choose to cooperate, he, for sure, will get longer in jail, but if he choose to betray he still has a chance to get lessen time in jail if his partner also choose to betray. If his partner choose to take the cooperate move, it's better off not choose the betray move because it'll be a great payoff. With same assumption, its hard to the first player to decide whether he has to betray his partner, or to cooperate, although in average it will tend to choose the betray move.

Considering that both player plays rationally and maximally, and also because of knowing that the second player will tend to choose the **betray** move (because still be a chance that they both will get lessen time in jail), looking at the c condition, the first player will also decide to take the **betray** move. With the same assumption, that the first player will tend to choose the **betray** move, the second player will also choose the **betray** move.

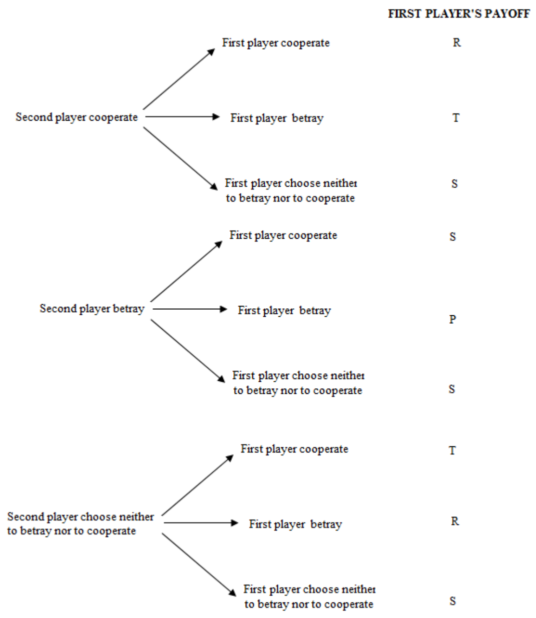
From those reason, we can see that for the asymmetric prisoner's dilemma problem, both player will also tend to choose betray each other than to cooperate because if they both choose to betray, it will give better payoff than if they both choose to cooperate.

2.3 Multiple Moves Prisoner's Dilemma

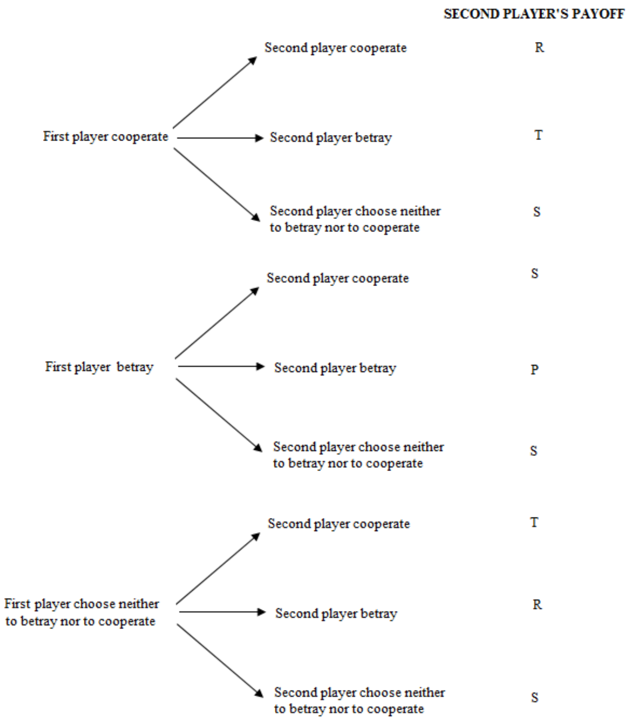
This prisoner's dilemma problem described by the following payoff matrix

	Cooperate	**Betray**	**Neither**
Cooperate	R,R	S,T	T,S
Betray	T,S	P,P	R,S
Neither	S,T	S,R	S,S

From the payoff matrix, the first player's payoff tree diagram can be shown as



The second player's payoff tree diagram can be shown as



Considering that the common condition is T>R>P>S. It will be hard to both player to decide. It will be easy to the first player to decide what move he must take if he assume that the second player will only choose to cooperate or betray him. With the same thinking with the classic dilemma's problem (either symmetric or asymmetric), the first player will tend to choose the betray move than to cooperate because it'll give a better payoff for him. Same with it, if the second player also assume that the first player will only choose to cooperate or betray him., the second player will tend to choose the betray move than to cooperate.

But it has a different condition here, both players can choose neither cooperate nor betray. If the first player assume that the second player will decide neither cooperate nor betray, then the second player will get T if he decide to take cooperate move, get R if he decide to take betray move, S get R if he decide to take neither betraying nor cooperating. If we assume that both player want a selfish outcome, they will tend to choose cooperate than betray when he assume that the other player will decide neither betraying nor cooperating because it will give a bigger payoff for the other player if he choose to cooperate.

But if we consider that both player play rationally and maximally, both player should choose neither betraying nor cooperating. If we look at the payoff matrix and the payoff tree diagram, they still can get the smallest payoff when they both choosing neither to take the betray move nor to take the cooperation move.

2.4 Multiple Players Prisoner's Dilemma

As shown before, for this type is R happens if all cooperate, P happens of all betray, and if some cooperate and some betray, it would pay S for the cooperators and T for the betrayers.

More generally, there is some social benefit B that each member can achieve if sufficiently many pay a cost C. Assuming that C is a negative number, the "temptation" here is to get the benefit without the cost, the "reward" is the benefit with the cost, the "punishment" is to get neither and the "sucker" payoff is to pay the cost without taking the benefit. So the payoffs are ordered B > (B+C) > 0 > C. If we take an example in the two-player game, it appears that all players will tend to choose **betray move and** achieve 0, **than the** cooperate **move** while preferring that if everyone decide to take **cooperate move** and obtain C+B.

The game labeled a many-person prisoner's dilemma requires that the payoff to each co-operator and betrayer increases strictly with the number of cooperators and that the sum of the payoffs to all parties increases with the number of cooperators (so that one party's switching from betrayal to cooperation always raises the sum). For hold j close to the threshold t for minimally effective cooperation, we can calculate the payoff with this calculation

- for every individual i, $B(i, j+1)+C(i, j+1)>B(i, j)+C(i, j)$ for $j>t$,
- for every individual i, $C(i, j+1)>C(i, j)$ for $j \leq t$, and
- $B(1, j+1)+C(1, j+1)+\ldots+B(j+1, j+1)+C(t+1, j+1)+B(j+2, j+1))+\ldots+B(n, j+1) > B(1, j)+C(1, j)+\ldots+B(j, j)+C(j, j)+B(j+1, j)+\ldots+B(n, j)$.

From the descriptions above, we can't see that in a multiple players prisoner's dilemma, it's better of choosing to keep taking the cooperate moves than taking the betray move. It's true that if the universal betrayal condition happen, it will give the best payoff (which is 0) to the player, but, the reality is there are so many players, and it hard to create the betrayal condition, especially if the player number is a big number. If some player decide to take the cooperate move, it will give such a great payoff to the player who keep taking the betray move. The more players decide to take the corporate move, the smaller payoff they get.

III. Conclusion

The prisoner's dilemma is a game model for analyzing about decision making which reflect the reality that happen in our daily life. For the classic two-player prisoner's dilemma problems, it's better off to both players deciding to take the betray move because it will lessen their time in jail. But, if they are given choice neither to betray nor to cooperate, it's better choosing neither betray nor cooperate, because if they decide to take the betray move or the cooperate move while another takes neither to betray nor cooperate, it will give bigger payoff. For the multiple players, it's better off choosing to keep taking the cooperate moves than taking the betray move because it's true) to the player because taking the betray move has a greater risk to get a greater payoff. With this explanation, we can use this reasoning when we are in the similar situation with this prisoner's dilemma simulation to choose the best choice which will give the best payoff.

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STATEMENT

With this I state that this paper is my own writing, not adaption, or translation from other's paper, and not plagiarism.

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