The Application of Modular Arithmetic in Determining a Leap Year

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***Abstract*—This paper discusses the application of modular arithmetic in determining a leap year. Generally, every year is 365 1/4 days. The 1/4 part adds up to a whole day every four years. Because of this, there is an extra day in a leap year. The extra day, called Leap Day, is on Feb 29. Leap year first used in the Julian calendar then refined again in the Georgian calendar.**

***Index Terms*—modular arithmetic, leap year, calendar system, algorithm, flowchart.**

I. Introduction

In mathematics, modular arithmetic (sometimes called clock arithmetic) is a system of arithmetic for integers, where numbers "wrap around" after they reach a certain value—the modulus[1]. Modular arithmetic was further advanced by Carl Friedrich Gauss in his book Disquisitiones Arithmeticae, published in 1801.

A familiar use of modular arithmetic is in the 12-hour clock, in which the day is divided into two 12 hour periods. If the time is 7:00 now, then 8 hours later it will be 3:00. Usual addition would suggest that the later time should be 7 + 8 = 15, but this is not the answer because clock time "wraps around" every 12 hours; there is no "15 o'clock". Likewise, if the clock starts at 12:00 (noon) and 21 hours elapse, then the time will be 9:00 the next day, rather than 33:00. Since the hour number starts over after it reaches 12, this is arithmetic modulo 12.

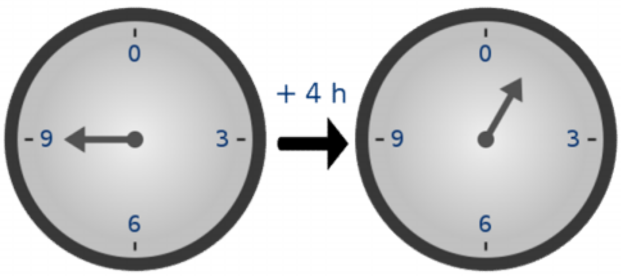

Fig. 1 Time-keeping on this clock uses arithmetic modulo 12

Operators used in modulo arithmetic is modulo[2]. Modulo operator, if used in integer division, will provide the rest of the division. This modulo operator is very useful in calculation of daily life, especially in determining leap years in the calendar system.

A calendar is a system of organizing days for social, religious, commercial, or administrative purposes[3]. This is done by giving names to periods of time, typically days, weeks, months, and years. The name given to each day is known as a date. Periods in a calendar (such as years and months) are usually, though not necessarily, synchronized with the cycle of the sun or the moon. Many civilizations and societies have devised a calendar, usually derived from other calendars on which they model their systems, suited to their particular needs.

Why is there a leap year? Generally, every year is 365 1/4 days. The 1/4 part adds up to a whole day every four years. Because of this, there is an extra day in a leap year. The extra day, called Leap Day, is on Feb 29.

II. Modulo Operation

In computing, the modulo operation finds the remainder of division of one number by another[3]. Given two positive numbers, a (the dividend) and n (the divisor), a modulo n (abbreviated as a mod n) can be thought of as the remainder, on division of a by n. For instance, the expression "5 mod 4" would evaluate to 1 because 5 divided by 4 leaves a remainder of 1, while "9 mod 3" would evaluate to 0 because the division of 9 by 3 leaves a remainder of 0, there is nothing to subtract from 9 after multiplying 3 times 3. When either a or n are negative, this naive definition breaks down and many programming languages differ in how these values are defined. Although typically performed with a and n both being integers, many computing systems allow other types of numeric operands.

There are various ways of defining a remainder, and computers and calculators have various ways of storing and representing numbers, so what exactly constitutes the result of a modulo operation depends on the programming language and/or the underlying hardware.

In nearly all computing systems, the quotient q and the remainder r satisfy

$$a = n \times q + r \text{, with } |r| < |n| \quad (1)$$

This means there are two possible choices for the remainder, one negative and the other positive, and there are also two possible choices for the quotient. Usually, in number theory, the positive remainder is always chosen, but programming languages choose depending on the language and the signs of a and n. However, Pascal and Algol68 do not satisfy these conditions for negative divisors, and some programming languages, such as C89, don't even define a result if either of n or a is negative. See the table for details. a modulo 0 is undefined in the majority of systems, although some do define it to be a.

Many implementations use truncated division where the quotient is defined by truncation q = trunc(a/n) and the remainder by r=a-n q. With this definition the quotient is rounded towards zero and the remainder has the same sign as the dividend.

Knuth described floored division where the quotient is defined by the floor function q=floor(a/n) and the remainder r is

$$r = a - n \left\lfloor \frac{a}{n} \right\rfloor . \quad (2)$$

Here the quotient rounds towards negative infinity and the remainder has the same sign as the divisor.

III. CALENDAR SYSTEM

A. Lunar and Solar Years

In Mesopotamia, where the Babylonians are the leading astronomers, the calendar is a simple lunar one[4]. So probably is the first Egyptian calendar. And a lunar calendar is still in use today in Islam. But such a calendar has one major disadvantage.

The length of a lunar month, from one new moon to the next, is 29.5 days. So twelve lunar months are 354 days, approximately 11 days short of a solar year. In a lunar year each of the twelve months slips steadily back through the seasons (as happens now with the Muslim calendar), returning to its original position only after 32 years.

In some lunar calendars an extra month is inserted from time to time to keep in step with the solar year. This happens in Mesopotamia and in republican Rome, and it remains the case today in the Jewish calendar.

But the Egyptian priests' observation of Sirius enables them to count the number of days in a solar year. They make it 365. They then very logically adjust the twelve months of the lunar year, making each of them 30 days long and adding 5 extra days at the end of the year. Compared to anybody else's calendar at the time this is very satisfactory. But there is a snag.

The priests cannot have failed to notice that every four years Sirius appears one day later. The reason is that the solar year is more exactly 365 days and 6 hours. The Egyptians make no adjustment for this, with the result that their calendar slides backwards through the seasons just like a lunar one but much more slowly. Instead of 32 years with the moon, it is 1460 years before Sirius rises again on the first day of the first month.

It is known from the records that in AD 139 Sirius rises on the first day of the first Egyptian month. This makes it certain that the Egyptian calendar is introduced one or two full cycles (1460 or 2920 years) earlier, either in 1321 or 2781 BC - with the earlier date considered more probable.

B. Julian and Mayan Calendars

The Roman calendar introduced by Julius Caesar, and subsequently known as the Julian calendar, gets far closer to the solar year than any predecessor. By the 1st century BC reform in Rome has become an evident necessity. The existing calendar is a lunar one with extra months slipped in from to time in an attempt to adjust it. In Caesar's time this calendar is three months out in relation to the seasons.

On the advice of Sosigenes, a learned astronomer from Alexandria, Caesar adds ninety days to the year 46 BC and starts a new calendar on 1 January 45.

Sosigenes advises Caesar that the length of the solar year is 365 days and six hours. The natural solution is to add a day every fourth year - introducing the concept of the leap year. The extra day is added to February, the shortest of the Roman months.

Spread through the Roman empire, and later throughout Christendom, this calendar proves very effective for many centuries. Only much later does a flaw yet again appear. The reason is that the solar year is not 365 days and 6 hours but 365 days, 5 hours, 48 minutes and 46 seconds. The difference amounts to only one day in 130 years. But over the span of history even that begins to show. Another adjustment will eventually be necessary.

While Julius Caesar is improving on the solar calendar of 365 days, a similar calendar has been independently arrived at on the other side of the Atlantic. Devised originally by the Olmecs of central America, it is perfected in about the 1st century AD by the Maya.

The Maya, establishing that there are 365 days in the year, divide them into 18 months of 20 days. Like the Egyptians (who have 12 months of 30 days), they complete the year by adding 5 extra days at the end - days which are considered to be extremely unlucky for any undertaking. An unusual aspect of the Mayan system is the Calendar Round, a 52-year cycle in which no two days have the same name.

C. The Working Week

Unlike the day, the month or the year, the week is an entirely artificial period of time. It is probably first made necessary by the demands of trade. Hunter-gatherers and primitive farmers have no need of such a concept, but commerce benefits from regularity. The original weeks are almost certainly the gaps between market days.

Weeks of this kind vary from four days among some

African tribes to ten days in the Inca civilization and in China. In ancient China a five-day week sets the working pattern for the Confucian civil service, every fifth day being a 'bath and hair-washing day'. Later this is extended to a ten-day week, with the three periods of each month known as the first, middle and last bath.

There are two possible sources for the seven-day week. One is the biblical creation story. From those times the Israelites have a week of this length, with the seventh day reserved for rest and worship (a pattern reflected in the Bible's account of creation).

The other and more likely source is Rome, where the equivalent of the modern week is adopted in about the 1st century AD - a time and a place where the Jewish tradition would have little influence. The number of days in the week derives probably, through astrology, from the seven known planets - which also provide the names of the days.

D. Jewish and Muslim Calendars

The Jewish calendar combines lunar and solar cycles. It is given its present form in AD 921 after a great debate between supporters of two slightly different systems.

In origin the calendar goes back to the captivity in Babylon, when the Jews adopt the Babylonians' calendar and their names for the months. They are lunar months of 30 or 29 days. In every second or third year an extra month of 30 days is added to keep the calendar in approximate step with the solar year. This constitutes a crucial difference between the Jewish and Muslim systems.

The Muslim calendar is the only one in widespread use to be based uncompromisingly on lunar months, with no adjustments to bring the years into balance with the solar cycle.

The twelve months are alternately 29 and 30 days long (the lunar cycle is approximately 29.5 days), giving a year of 354 days. There are two significant results. Muslim months bear no relation to the seasons, and Muslim years do not coincide with those of other chronologies. There are about 103 lunar years in a solar century. By the millennium there will have been 1421 lunar years but only 1378 solar years from the start of Muslim chronology in AH 1 or AD 622. The year AH 1421 will be AD 2000.

E. Gregorian Calendar

By the 16th century the seemingly minor error in the Julian calendar (estimating the solar year to be 11 minutes and 14 seconds shorter than it actually is) has accumulated to a ten-day discrepancy between the calendar and reality. It is most noticeable on occasions such as the equinox, now occuring ten days earlier than the correct calendar dates of March 21 and September 23.

Pope Gregory XIII employs a German Jesuit and astronomer, Christopher Clavius, to find a solution. Calculating that the error amounts to three days in 400 years, Clavius suggests an ingenious adjustment.

His proposal, which becomes the basis of the calendar known after the commissioning pope as Gregorian, is that century years (or those ending in '00') should only be leap years if divisible by 400. This eliminates three leap years in every four centuries and neatly solves the problem. The result, in the centuries since the reform, is that 1600 and 2000 are normal leap years, but the intervening 1700, 1800 and 1900 do not include February 29.

Gregory puts the proposal into immediate effect in the papal states, announcing that the day after October 4 in 1582 will be October 15 - thus saving the lost ten days.

The pope's lead is followed in the same year by Spain, Portugal, France and most Italian states. The German-speaking Roman Catholic states comply in 1583.

Other Christian realms drag their feet on the issue, reluctant to admit that the pope in Rome has a point. The Lutheran states of Germany change in 1700. Great Britain delays until 1752, by which time the gap is eleven days. Some of the British prove exceptionally dim over the issue, fearing that their lives are being shortened and in places even rioting for the return of the missing days. Imperial Russia never makes the change; it is introduced after the revolution, in 1918. (Potentially confusing dates, near the change-over years, are identified by historians with the codes OS or Old Style for the Julian version and NS or New Style for the Gregorian equivalent.)

More precise measurements in the 20th century have introduced a further refinement of the Gregorian calendar, though not one of immediate significance. As adjusted for pope Gregory, the present system adds one day in every 3,323 years. The accepted solution is that years divisible by 4000 will not be leap years.

February 29 will therefore be dropped unexpectedly in 2000 years' time. In AD 4000, even though the year is divisible by 400, March 1 will follow February 28 in the normal way. Julius Caesar and Sosigenes would no doubt be impressed by this ultimate refinement of their system, making it accurate to within one day in 20,000 years.

Fig. 2 The difference between the Gregorian calendar and the seasons

IV. Leap Year

Leap year first used in the Julian calendar then refined again in the Georgian calendar. Leap years are only used on both the calendar system. There are simple algorithms to determine whether a year including leap year or not using modulo operation as follows.

A. Leap Year on Julian Calendar

```
function IsLeapYear1 (y:integer)→ boolean
{true if y is a leap year}

KAMUS LOKAL

ALGORITMA
   if y mod 4 = 0 then
        → true
   else
        → false
```

Algorithm 3.1 Algorithm for determining leap year on Julian Calendar

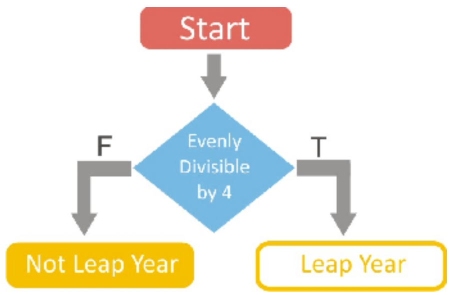

Fig. 3.1 Flowchart for determining leap year on Julian Calendar

B. Leap Year on Georgian Calendar

- Before 20th century

```
function IsLeapYear2 (y:integer)→ boolean
{true if y is a leap year}

KAMUS LOKAL

ALGORITMA
   if (y mod 4 = 0) or ((y mod 100 = 0) and
   (y mod 400 = 0)) then
        → true
   else
        → false
```

Algorithm 3.2 Algorithm for determining leap year on Gregorian Calendar before 20th century

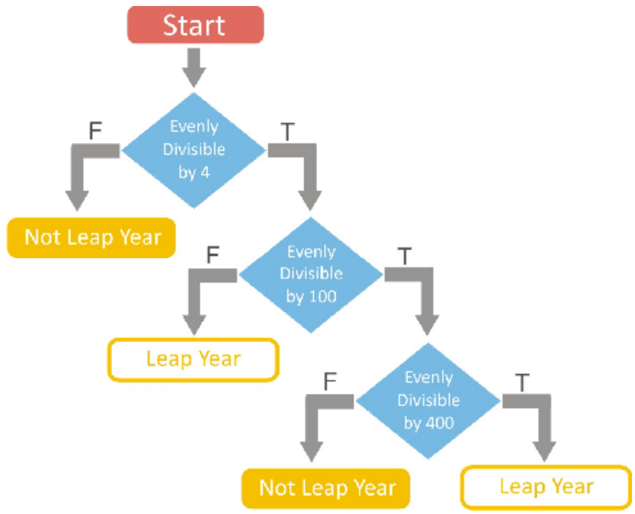

Fig. 3.2 Flowchart for determining leap year on Gregorian Calendar before 20th century

- After 20th century

```
function IsLeapYear3 (y:integer)→ boolean
{true if y is a leap year}

KAMUS LOKAL

ALGORITMA
   if (y mod 4 = 0) or ((y mod 100 = 0) and
   ((y mod 400 = 0) and (y mod 4000 ≠ 0))) then
        → true
   else
        → false
```

Algorithm 3.3 Algorithm for determining leap year on Gregorian Calendar after 20th century

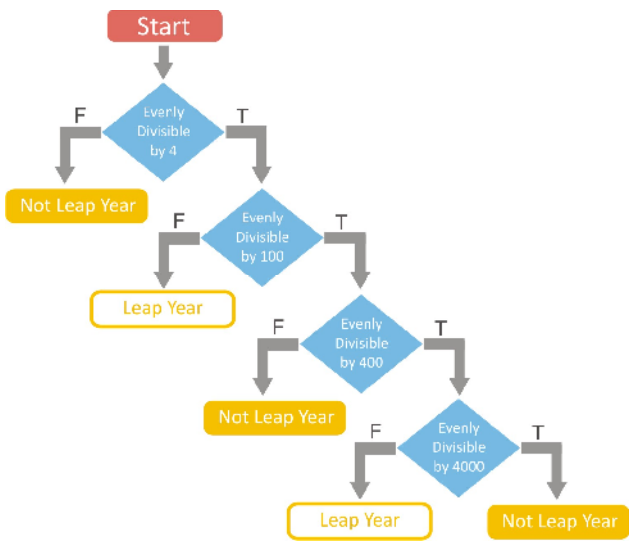

Fig. 3.3 Flowchart for determining leap year on Gregorian Calendar after 20th century

V. Conclusion

- The Gregorian calendar system is a calendar system based on solar year that "most prefect" compared to other calendar systems.
- Leap year calculation method is a method that uses the concept of modular arithmetic.

References

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Pernyataan

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

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