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**Seri bahan kuliah Algeo 24**

Bahan Kuliah IF2123 Aljabar Linier dan Geometri

# Dekomposisi QR

Program Studi Teknik Informatika
Sekolah Teknik Elektro dan Informatika
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# Dekomposisi QR

- Dekomposisi QR adalah memfaktorkan matriks berukuran m x n menjadi hasil kali matriks ortogonal dan matriks segitiga.

$$A = QR$$

*Q* = matriks ortonormal, *R* = matriks segitiga atas

- *Q* adalah matriks ortonormal (sekaligus ortogonal) sedemikian sehingga $QQ^T = I$.

- Ingatlah kembali defenisi matriks orthogonal. Matriks orthogonal adalah matriks yang setiap kolomnya adalah vektor sedemikian sehingga hasil kali titik setiap vektor dengan vektor lainnya = 0. Jika setiap vektor merupakan vektor satuan, maka disebut matriks ortonormal.

- Contoh:

$$
\underset{A}{\begin{pmatrix} 2.5 & 1.1 & 0.3 \\ 2.2 & 1.9 & 0.4 \\ 1.8 & 0.1 & 0.3 \end{pmatrix}} = \underset{Q}{\begin{pmatrix} -0.7 & 0.1 & -0.7 \\ -0.6 & -0.7 & 0.4 \\ -0.5 & 0.7 & 0.5 \end{pmatrix}} \underset{R}{\begin{pmatrix} -3.8 & -1.9 & -0.6 \\ 0. & -1.1 & 0. \\ 0. & 0. & 0.1 \end{pmatrix}}
$$

- Jika A matriks non-singular, maka dekomposisi A menghasilkan Q dan R yang unik.

- Ada beberapa metode untuk menghitung dekomposisi QR. Salah satu metode tersebut adalah proses Gram-Schmidt.

# Metode Gram-Schmidt

- Tinjau matriks A berukuran m x n. Setiap kolom pada matriks A dipandang sebagai vektor-vektor $a_1$, $a_2$, ..., $a_n$:

$$A = \left[ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n \right].$$

- Hitung $u_1$, $u_2$, ..., $u_{k+1}$ dan $e_1$, $e_2$, ..., $e_{k+1}$ sebagai berikut:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{a}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \\
\mathbf{u}_2 &= \mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{e}_1)\mathbf{e}_1, & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{||\mathbf{u}_2||}. \\
\mathbf{u}_{k+1} &= \mathbf{a}_{k+1} - (\mathbf{a}_{k+1} \cdot \mathbf{e}_1)\mathbf{e}_1 - \cdots - (\mathbf{a}_{k+1} \cdot \mathbf{e}_k)\mathbf{e}_k, & \mathbf{e}_{k+1} &= \frac{\mathbf{u}_{k+1}}{||\mathbf{u}_{k+1}||}.
\end{aligned}$$

- Maka, hasil faktorisasi A adalah:

$$A = \left[\, \mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n \,\right] = \left[\, \mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n \,\right] \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{e}_1 & \mathbf{a}_2 \cdot \mathbf{e}_1 & \cdots & \mathbf{a}_n \cdot \mathbf{e}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{e}_2 & \cdots & \mathbf{a}_n \cdot \mathbf{e}_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{a}_n \cdot \mathbf{e}_n \end{bmatrix} = QR.$$

- Untuk matriks A berukuran 3 x 3, hasil faktorisasinya adalah:

$$\overset{\mathbf{A}}{\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix}} = \underbrace{\overset{\mathbf{Q}}{\begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix}}}_{\text{Orthogonal Unit vectors}} \underbrace{\overset{\mathbf{R}}{\begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{a}_1 & \mathbf{e}_1 \cdot \mathbf{a}_2 & \mathbf{e}_1 \cdot \mathbf{a}_3 \\ 0 & \mathbf{e}_2 \cdot \mathbf{a}_2 & \mathbf{e}_2 \cdot \mathbf{a}_3 \\ 0 & 0 & \mathbf{e}_3 \cdot \mathbf{a}_3 \end{bmatrix}}}_{\text{Upper Diagonal Matrix}}$$

**Contoh 1**: Tinjau matriks A berikut, $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Vektor-vektor kolomnya adalah $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, dan $a_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, atau dalam notasi baris adalah

$$\mathbf{a}_1 = (1,1,0)^T,\ \mathbf{a}_2 = (1,0,1)^T,\ \mathbf{a}_3 = (0,1,1)^T.$$

Lakukan metode Gram-Schmidt untuk mendekomposisi matriks menjadi Q dan R sebagai berikut:

$$
\begin{aligned}
\mathbf{u}_1 &= \mathbf{a}_1 = (1,1,0), \\
\mathbf{e}_1 &= \frac{\mathbf{u}_1}{||\mathbf{u}_1||} = \frac{1}{\sqrt{2}}(1,1,0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \\
\\
\mathbf{u}_2 &= \mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{e}_1)\mathbf{e}_1 = (1,0,1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \left(\frac{1}{2}, -\frac{1}{2}, 1\right), \\
\mathbf{e}_2 &= \frac{\mathbf{u}_2}{||\mathbf{u}_2||} = \frac{1}{\sqrt{3/2}}\left(\frac{1}{2}, -\frac{1}{2}, 1\right) = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \\
\\
\mathbf{u}_3 &= \mathbf{a}_3 - (\mathbf{a}_3 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{a}_3 \cdot \mathbf{e}_2)\mathbf{e}_2 \\
&= (0,1,1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \\
\mathbf{e}_3 &= \frac{\mathbf{u}_3}{||\mathbf{u}_3||} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).
\end{aligned}
$$

Selanjutnya hitung:

$$a_1 \cdot e_1 = (1, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = (1)(\frac{1}{\sqrt{2}}) + (1)(\frac{1}{\sqrt{2}}) + (0)(0) = \frac{2}{\sqrt{2}}$$

$$a_2 \cdot e_1 = (1, 0, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = (1)(\frac{1}{\sqrt{2}}) + (0)(\frac{1}{\sqrt{2}}) + (1)(0) = \frac{1}{\sqrt{2}}$$

$$a_3 \cdot e_1 = (0, 1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = (0)(\frac{1}{\sqrt{2}}) + (1)(\frac{1}{\sqrt{2}}) + (1)(0) = \frac{1}{\sqrt{2}}$$

dan seterusnya, hitung $a_2 \cdot e_2$ , $a_3 \cdot e_2$ , dan $a_3 \cdot e_3$ , yang hasilnya adalah sbb:

$$a_2 \cdot e_2 = \frac{3}{\sqrt{6}}\ , \quad a_3 \cdot e_2 = \frac{1}{\sqrt{6}}\ , \quad a_3 \cdot e_3 = \frac{2}{\sqrt{3}}$$

# Hasil dekomposisi QR:

$$Q = \left[ \mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n \right] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

$$R = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{e}_1 & \mathbf{a}_2 \cdot \mathbf{e}_1 & \mathbf{a}_3 \cdot \mathbf{e}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{e}_2 & \mathbf{a}_3 \cdot \mathbf{e}_2 \\ 0 & 0 & \mathbf{a}_3 \cdot \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}.$$

**Contoh 2:** Dekomposisi matriks A berukuran 4 x 3 berikut menjadi Q dan R:

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

**Penyelesaian:** n = 3 (jumlah kolom), maka jawaban akan berbentuk sebagai berikut:

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{a}_1 & \mathbf{e}_1 \cdot \mathbf{a}_2 & \mathbf{e}_1 \cdot \mathbf{a}_3 \\ 0 & \mathbf{e}_2 \cdot \mathbf{a}_2 & \mathbf{e}_2 \cdot \mathbf{a}_3 \\ 0 & 0 & \mathbf{e}_3 \cdot \mathbf{a}_3 \end{bmatrix}$$

$$\mathbf{u1} = \mathbf{a1} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{e1} = \frac{\mathbf{u1}}{\|\mathbf{u1}\|} = \frac{(-1,1,-1,1)}{2} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\mathbf{u2} = \mathbf{a2} - (\mathbf{a2} \cdot \mathbf{e1})\mathbf{e1} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{e2} = \frac{\mathbf{u2}}{\|\mathbf{u2}\|} = \frac{(1,1,1,1)}{2} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\mathbf{u3} = \mathbf{a3} - (\mathbf{a3} \cdot \mathbf{e1})\mathbf{e1} - (\mathbf{a3} \cdot \mathbf{e2})\mathbf{e2} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} - 2 \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} - 8 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{e3} = \frac{\mathbf{u3}}{\|\mathbf{u3}\|} = \frac{(-2,-2,2,2)}{4} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Selanjutnya hitung:

$a_1 \cdot e_1 = (-1, 1, -1, 1) \cdot (-½, ½, -½, ½) = (-1)(-\frac{1}{2}) + (1)(\frac{1}{2}) + (-1)(-\frac{1}{2}) + (1)(\frac{1}{2}) = 2$

$a_2 \cdot e_1 = 4$

$a_3 \cdot e_1 = 2$

dan seterusnya, hitung $a_2 \cdot e_2$ , $a_3 \cdot e_2$ , dan $a_3 \cdot e_3$ , yang hasilnya adalah sbb:

$a_2 \cdot e_2 = 2$ , $a_3 \cdot e_2 = 8$ , $a_3 \cdot e_3 = 4$

# Hasil akhir:

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

# Cara lain menghitung matriks R

Perhatikan:

$$A = QR$$

Kalikan kedua ruas dengan $Q^T$:

$$\begin{aligned} Q^T A &= Q^T QR \\ &= I\,R \qquad \text{(karena } Q^T Q = I\text{)} \\ &= R \end{aligned}$$

Jadi, $R = Q^T A$

- Pada contoh sebelumnya sudah diperoleh:

$$Q = \begin{bmatrix} -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix}$$

maka

$$R = Q^T A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

## Determinants

Let $A$ be $n \times n$. Then $Q$ and $R$ are both $n \times n$ as well.[1] Since $Q$ is orthonormal and $R$ is upper-triangular,

$$\det(Q) = \pm 1 \qquad \text{and} \qquad \det(R) = \prod_{i=1}^{n} r_{i,i}.$$

Then since $\det(AB) = \det(A)\det(B)$,

$$|\det(A)| = |\det(QR)| = |\det(Q)\det(R)| = |\det(Q)|\,|\det(R)| = \left|\prod_{i=1}^{n} r_{i,i}\right|. \tag{3.1}$$

Sumber: L. Vandenberghe, ECE133A (Fall 2024), QR factorization

- Dekomposisi QR tidak unik, tapi sangat dekat ke unik, cukup dengan mengubah tanda (plus/min) pada **e1**, **e2**, dst, seperti contoh di bawah ini:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} -\sqrt{2} & -1/\sqrt{2} \\ 0 & \sqrt{6} \end{bmatrix} =$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & -\sqrt{6} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} -\sqrt{2} & -1/\sqrt{2} \\ 0 & -\sqrt{6} \end{bmatrix}.$$

semuanya adalah hasil dekomposisi QR dari matriks $\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$

# Dekomposisi QR di dalam Python

```
import pprint
```

```
import numpy as np
```

```
matrix1 = np.array([[1, 2, 3], [3, 4, 5]])
```

```
print (matrix1)
```

```
[[1 2 3]
 [3 4 5]]
```

```
q, r = np.linalg.qr(matrix1)
```

```
print('\nQ:\n', q)
```

```

Q:
 [[-0.31622777 -0.9486833 ]
 [-0.9486833   0.31622777]]
```

```
print('\nR:\n', r)
```

```

R:
 [[-3.16227766 -4.42718872 -5.69209979]
 [ 0.         -0.63245553 -1.26491106]]
```

# Latihan

Dekomposisi matriks berikut dengan QR decomposition

(a) $$\begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

# Menyelesaikan SPL dengan Dekomposisi QR

- Tinjau SPL:

$$Ax = b$$

- Ganti A dengan QR:

$$QRx = b$$

- Kalikan kedua ruas dengan $Q^T$:

$$Q^TQRx = Q^Tb$$

$$I\,Rx = Q^Tb$$

$$Rx = Q^Tb$$

- Oleh k arena R adalah matriks segitiga atas, maka solusi SPL dapat dicari dengan teknik penyulihan mundur.

**Contoh:** Selesaikan SPL Ax = b dengan A dan b sebagi berikut, menggunakan dekomposisi QR

$$A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

**Jawaban:** Hasil dekomposisi QR matriks A adalah:

$$Q = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \approx \begin{bmatrix} -0.707106781186548 & 0.707106781186548 \\ 0.707106781186548 & 0.707106781186548 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 4\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 1.414213562373095 & 1.414213562373095 \\ 0 & 5.65685424949238 \end{bmatrix}$$

Solusi SPL adalah sbb:

$Rx = Q^T b$

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 4\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Lakukan penyulihan mundur:

Persamaan baris ke-2: $4\sqrt{2}\,x_2 = \frac{\sqrt{2}}{2} \rightarrow x_2 = \frac{\frac{\sqrt{2}}{2}}{4\sqrt{2}} = \frac{1}{8}$

Persamaan baris ke-1: $\sqrt{2}\,x_1 + \sqrt{2}\,x_2 = -\frac{\sqrt{2}}{2} \rightarrow x_1 = \frac{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8}}{\sqrt{2}} = \frac{\frac{-5\sqrt{2}}{8}}{\sqrt{2}} = \frac{-5}{8}$

# Latihan

Selesaikan SPL $Ax = b$ berikut dengan metode dekomposisi QR:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 2 & -2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

# Referensi:


1. Igor Yanovsky (Math 151B TA), QR Decomposition with Gram-Schmidt
2. L. Vandenberghe, ECE133A (Fall 2024), QR factorization