Seri bahan kuliah Algeo #18 - 2023

Ruang Vektor Umum (bagian 4) dan Transformasi Linier

Bahan kuliah IF2123 Aljabar Linier dan Geometri

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Sumber:

Howard Anton & Chris Rores, *Elementary Linear Algebra, 10^{th} Edition*

Transformasi Linier

- Transformasi = fungsi =pemetaan (*mapping*)

DEFINISI 1: Misalkan V dan W adalah ruang vektor. Transformasi yang memetakan ruang vektor V ke ruang vektor W ditulis sebagai

$$T : V \rightarrow W$$

V adalah daerah asal (domain) transformasi T dan W adalah daerah hasil transformasi (kodomain) fungsi. Jika V = W, maka T dinamakan **operator** pada V.

- Jika $\mathbf{v} \in V$ dan $\mathbf{w} \in W$, maka

$$\mathbf{w} = T(\mathbf{v})$$

Contoh 1: Misalkan T : $R^3 \rightarrow R^3$ didefinisikan sebagai berikut:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + 5x_2 \\ x_3 \end{bmatrix}$$

Tentukan bayangan vektor **v** = (3, 2, 0).

Jawaban:

$$T\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 + 2(2) + 0 \\ 3 + 5(2) \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \\ 0 \end{bmatrix}$$

Jadi, bayangan vektor (3, 2, 0) adalah (7, 13, 0).

DEFINISI 2: Misalkan V dan W adalah ruang vektor. Transformasi

$$T : V \rightarrow W$$

dinamakan **transformasi linier** jika untuk semua **u** dan **v** di dalam **V** dan *k* sebuah skalar berlaku:

(1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(2) $T(k\mathbf{u}) = kT(\mathbf{u})$

Jika V = W, maka T dinamakan **operator** linier pada V.

Contoh 2: Diberikan fungsi T : $R^2 \rightarrow R^2$ yang dalam hal ini T(x,y) = (2x, y), maka akan ditunjukkan bahwa T adalah transformasi linier.

Misalkan **u** dan **v** adalah dua buah vektor di R^2, **u** = (u_1, u_2) dan **v** = (v_1, v_2).

(1) T(**u** + **v**) = $T(u_1 + v_1, u_2 + v_2) = (2(u_1 + v_1), u_2 + v_2) = (2u_1 + 2v_1, u_2 + v_2)$

$$= \begin{bmatrix} 2u_1 + 2v_1 \\ u_2 + v_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 \\ v_2 \end{bmatrix} = T(\mathbf{u}) + T(\mathbf{v})$$

(2) T(*k***u**) = $T(ku_1, ku_2) = (2ku_1, ku_2)$

$$= \begin{bmatrix} 2ku_1 \\ ku_2 \end{bmatrix} = k \begin{bmatrix} 2u_1 \\ u_2 \end{bmatrix} = kT(\mathbf{u})$$

Karena T(**u** + **v**) = T(**u**) + T(**v**) dan T(*k***u**) = *k*T(**u**), maka T adalah transformasi linier

Contoh 3: Diberikan fungsi T : $R^2 \rightarrow R^2$ yang dalam hal ini T(x,y) = (x, y + 1), maka akan ditunjukkan bahwa T bukan transformasi linier.

Misalkan **u** dan **v** adalah dua buah vektor di R^2, **u** = (u_1, u_2) dan **v** = (v_1, v_2).

(1) T(**u** + **v**) = T($u_1 + v_1$, $u_2 + v_2$) = ($u_1 + v_1$, $u_2 + v_2 + 1$)

$$= \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 + 1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 + 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = T(\mathbf{u}) + ?$$

Karena T(**u** + **v**) $\neq$ T(**u**) + T(**v**) maka T bukan transformasi linier

- Jika T : V $\rightarrow$ W adalah transformasi linier, $\mathbf{v_1}$ dan $\mathbf{v_2} \in \mathbf{V}$, dan k_1 dan k_2 adalah skalar maka

$$T(k_1\mathbf{v_1} + k_2\mathbf{v}_2) = T(k_1\mathbf{v_1}) + T(k_2\mathbf{v}_2) = k_1T(\mathbf{v_1}) + k_2T(\mathbf{v}_2)$$

- Secara umum, jika $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in$ ..**V**, dan $k_1, k_2, \ldots, k_n$ adalah skalar maka

$$T(k_1\mathbf{v_1} + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n) = k_1T(\mathbf{v_1}) + k_2T(\mathbf{v}_2) + \ldots + k_nT(\mathbf{v}_n)$$

Teorema 1: Jika T : V $\rightarrow$ W adalah transformasi linier, maka

(1) T(0) = 0

(2) T(–**v**) = –T(**v**)

(3) T(**u** – **v**) = T(**u**) – T(**v**)

Transformasi Matriks dari R^n ke R^m

- Jika $V = R^n$ dan $W = R^m$, maka

$$T : R^n \rightarrow R^m$$

- Jika $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n$ dan $\mathbf{w} = (w_1, w_2, \ldots, w_m) \in R^m$ maka

$$(w_1, w_2, \ldots, w_m) = T(x_1, x_2, \ldots, x_n)$$

yang dalam hal ini,

$$w_1 = f_1(x_1, x_2, \ldots, x_n)$$
$$w_2 = f_2(x_1, x_2, \ldots, x_n)$$
$$\ldots$$
$$w_m = f_m(x_1, x_2, \ldots, x_n)$$

- Jika $f_1, f_2, \dots, f_m$ linier maka

$$w_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$w_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$

…

$$w_m = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n$$

yang dapat ditulis dengan notasi matriks:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

atau dalam bentuk ringkas

$$\mathbf{w} = A\mathbf{x}$$

A disebut **matriks standard** transformasi sedangkan transformasi T dinamakan **transformasi matriks**, sehingga $\mathbf{w} = A\mathbf{x}$ dapat ditulis sebagai

$$\mathbf{w} = T_A(\mathbf{x})$$

Contoh 4. Tranformasi matriks T : $R^4 \rightarrow R^3$ didefinisikan sebagai berikut

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Matriks standard transformasi adalah

$$A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}$$

Jika **x** = (1, –3, 0, 2), maka hasil transformasi T adalah

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$$

Jadi, **w** = (1, 3, 8)

Teorema. Untuk setiap matriks A, transformasi matriks $T_A : R^n \rightarrow R^m$ memiliki sifat-sifat sebagai berikut untuk semua vektor **u** dan **v** di dalam R^n dan untuk setiap skalar *k*:

(a) $T_A(\mathbf{0}) = \mathbf{0}$

(b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$

(c) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$

(d) $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

Latihan (Kuis 2022)

Misalkan

$$\left\{\overrightarrow{v_1} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \overrightarrow{v_2} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \overrightarrow{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$ adalah basis bagi R^3.

$T: R^3 \rightarrow P_1$ Transformasi linear didefinisikan $T(\overrightarrow{v_i}) = A\overrightarrow{v_i} = p_i$ untuk setiap $i = 1,2,3$.

Jika

$p_1 = 1 - x;\ p_2 = 1;\ p_3 = 2x$

a. (NILAI : 20)

Carilah matriks transformasi T

b. (NILAI : 15)

Tentukan hasil transformasi dari $T\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Jawaban: P_1 artinya ruang polinom derajat 1, setiap polinom ditulis dalam bentuk $p(x) = a_0 + a_1x$

a)

Jadi, $\mathbf{p_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{p_3} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Karena $A\mathbf{v_i} = \mathbf{p_i}$, maka,

$$A \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Jadi, matriks transformasi T adalah

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix}$$

b)

$$T\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 + x$$

Prosedur Menemukan Matriks Standard

Step 1: Tentukan bayangan dari semua vektor basis standard $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$ di R^n , yaitu

$$T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$$

dalam bentuk kolom.

Step 2: Konstruksi matriks yang memiliki bayangan-bayangan hasil dari Step1 sebagai kolom-kolom yang berurutan. Matriks tersebut adalah matriks standard untuk transformasi.

- Secara umum, jika

$$T(\mathbf{e_1}) = \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{m1} \end{bmatrix}, T(\mathbf{e_2}) = \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots., T(\mathbf{e_n}) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

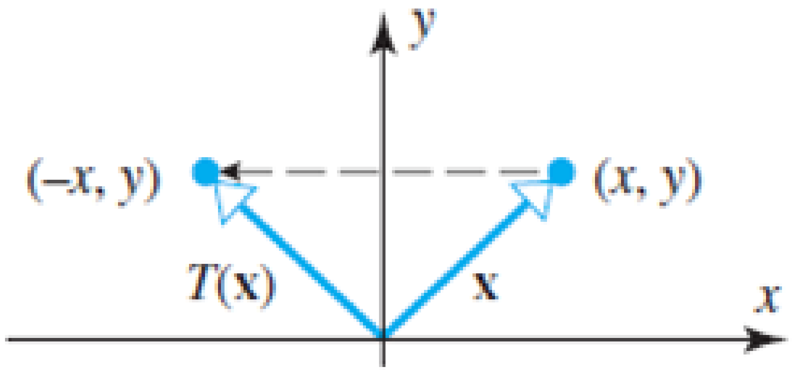
maka

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & & \uparrow \\ T(\mathbf{e_1}) & T(\mathbf{e_2}) & \ldots & T(\mathbf{e_n}) \end{matrix}$$

adalah matriks standard untuk $T : R^n \rightarrow R^m$

Contoh 5: Tentukan matriks standard untuk pencerminan vektor di R^2 terhadap sumbu-Y.



Pencerminan vektor **x** = (x, y) terhadap sumbu-Y
Hasil pencerminan adalah **x'** = T(**x**) = (–x , y)

$\mathbf{e}_1$ = (1, 0) → T($\mathbf{e}_1$) = (–1, 0)
$\mathbf{e}_2$ = (0, 1) → T($\mathbf{e}_2$) = (0, 1)

Matriks standard: A = $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Contoh 6: Carilah matriks standard dari transformasi T : $R^3 \rightarrow R^3$ yang didefinisikan sebagai berikut:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + 5x_2 \\ x_3 \end{bmatrix}$$

Lalu tentukan bayangan vektor **v** = (3, 2, 0).

Jawaban:

$\mathbf{e_1}$ = (1, 0, 0) $\rightarrow$ T($\mathbf{e_1}$) = $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{pmatrix} 1 + 2(0) + 0 \\ 1 + 5(0) \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\mathbf{e_2}$ = (0, 1, 0) $\rightarrow$ T($\mathbf{e_2}$) = $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{pmatrix} 0 + 2(1) + 0 \\ 0 + 5(1) \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$

$\mathbf{e_3}$ = (0, 0, 1) $\rightarrow$ T($\mathbf{e_3}$) = T$\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{pmatrix}0+2(0)+1\\0+5(0)\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$

Matriks standard adalah

$$A = \begin{bmatrix}1 & 2 & 1\\1 & 5 & 0\\0 & 0 & 1\end{bmatrix}$$

Jika **v** = (3, 2, 0), maka bayangan **v** adalah **w**,

$$\mathbf{w} = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}1 & 2 & 1\\1 & 5 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\2\\0\end{bmatrix} = \begin{bmatrix}7\\13\\0\end{bmatrix}$$

Table 1

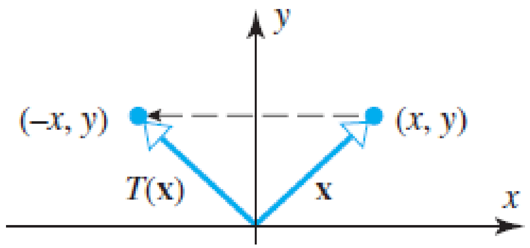
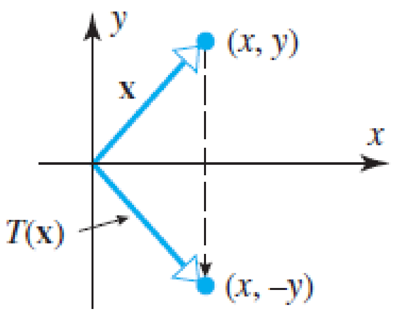
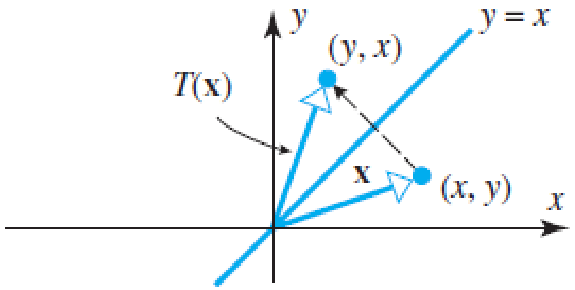
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Table 2

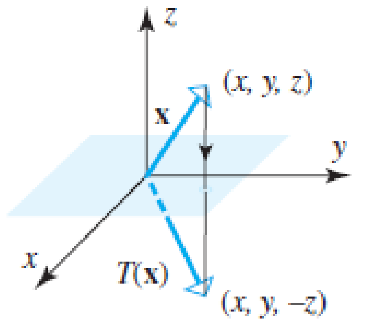
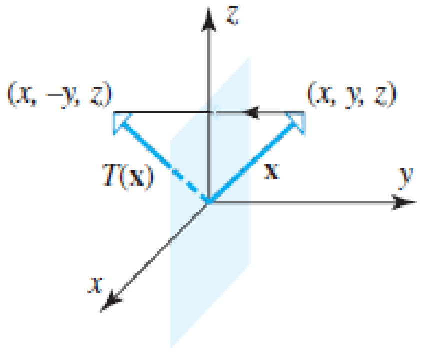
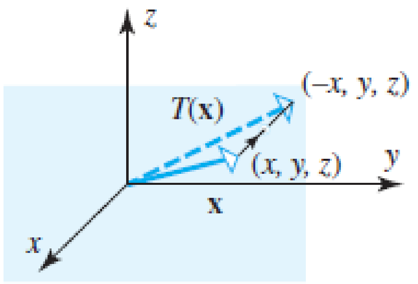
Operator	Illustration	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Table 3

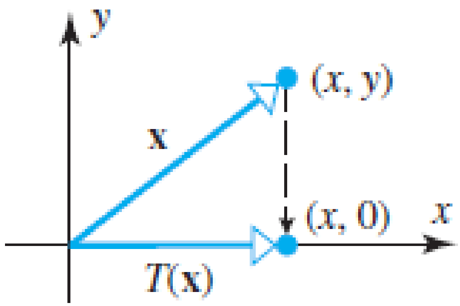
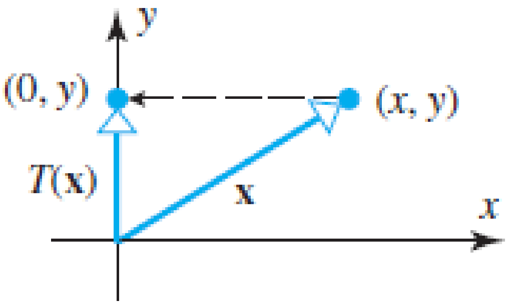
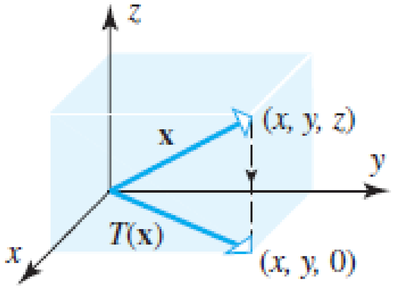
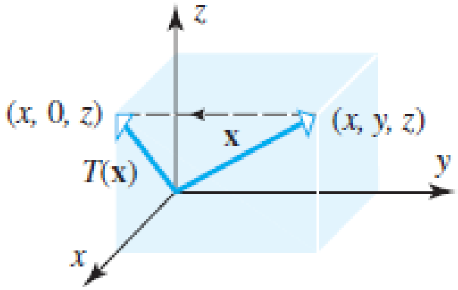
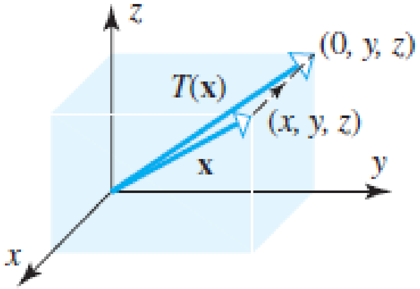
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection on the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection on the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Table 4

Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Orthogonal projection on the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection on the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection on the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operator Rotasi

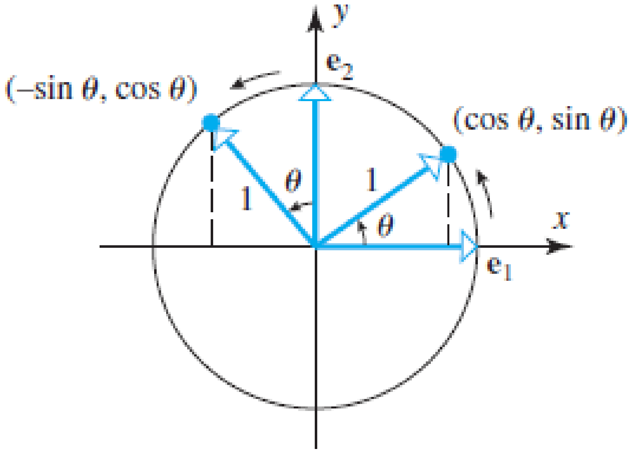


Table 5

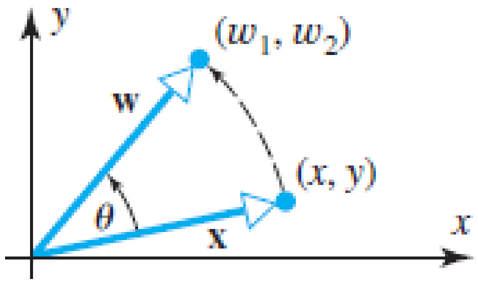
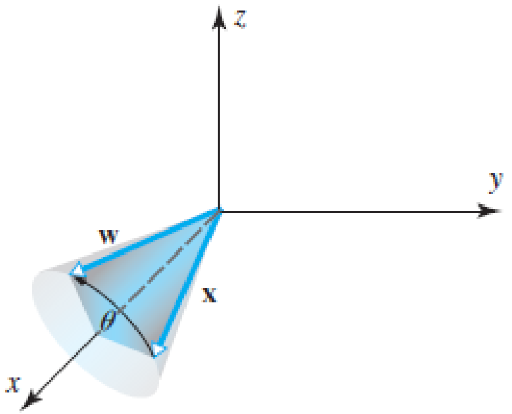
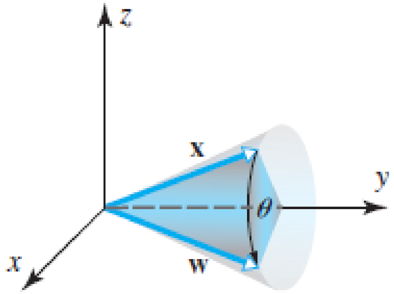
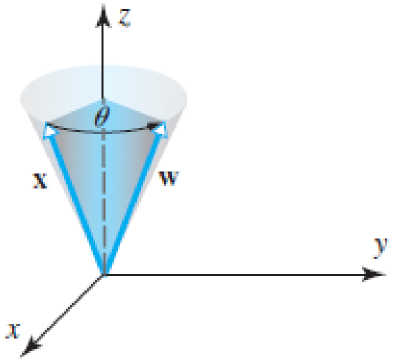
Operator	**Illustration**	**Rotation Equations**	**Standard Matrix**
Rotation through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Table 6

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Dilatasi dan Kontraksi

Table 7

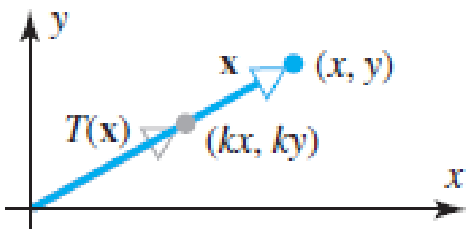
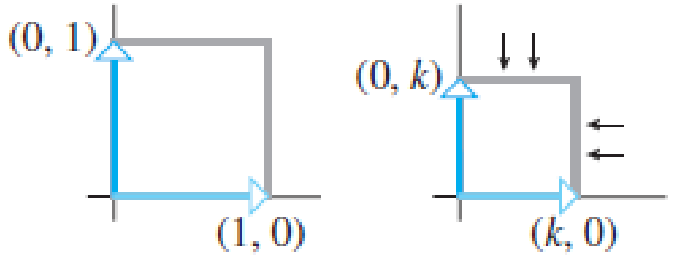
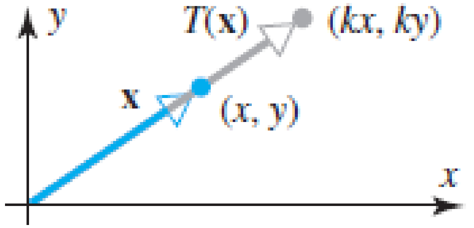
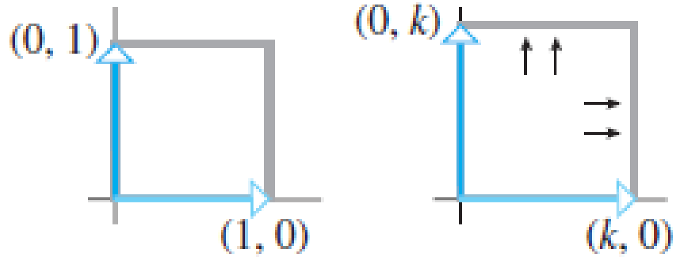
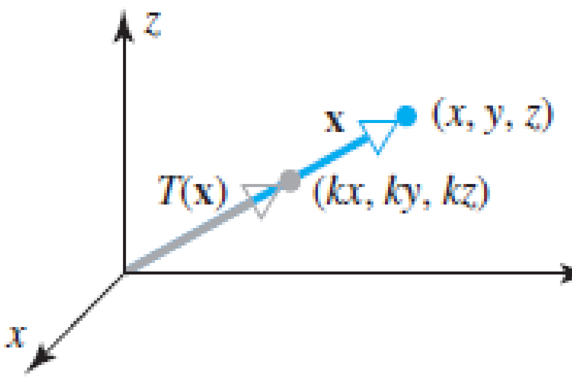
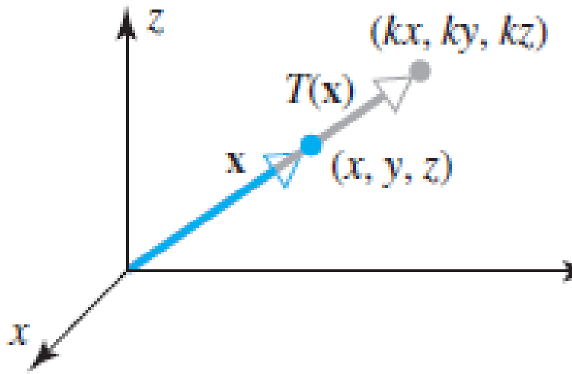
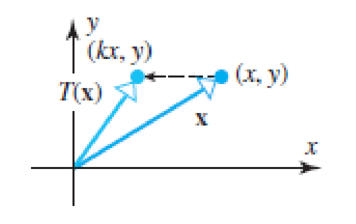
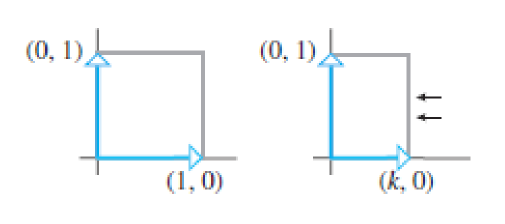
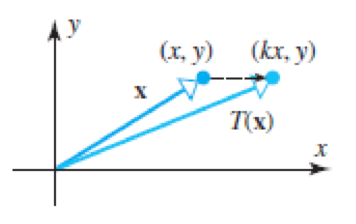
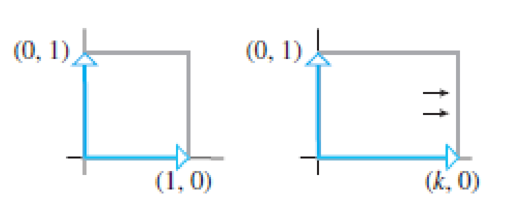
Operator	Illustration $T(x, y) = (kx, ky)$	Effect on the Standard Basis	Standard Matrix
Contraction with factor k on R^2 $(0 \leq k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k on R^2 $(k > 1)$			

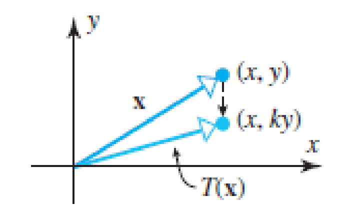
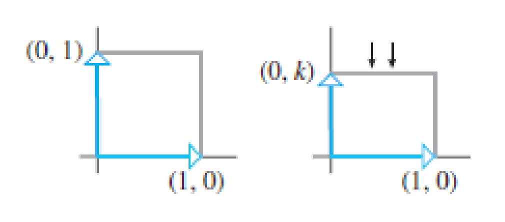
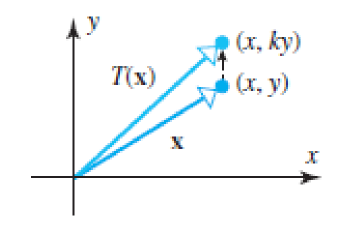
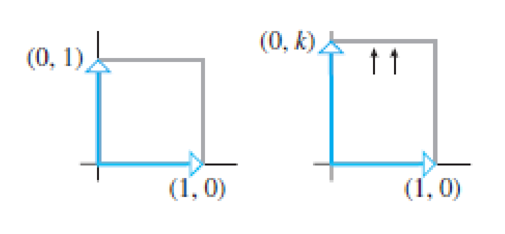
Table 8

<table>
<tr><th>Operator</th><th>Illustration
$T(x, y, z) = (kx, ky, kz)$</th><th>Standard Matrix</th></tr>
<tr><td>Contraction with factor k on R^3
$(0 \leq k \leq 1)$</td><td></td><td rowspan="2">$$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$</td></tr>
<tr><td>Dilation with factor k on R^3
$(k \geq 1)$</td><td></td></tr>
</table>

Ekspansi dan Kompresi

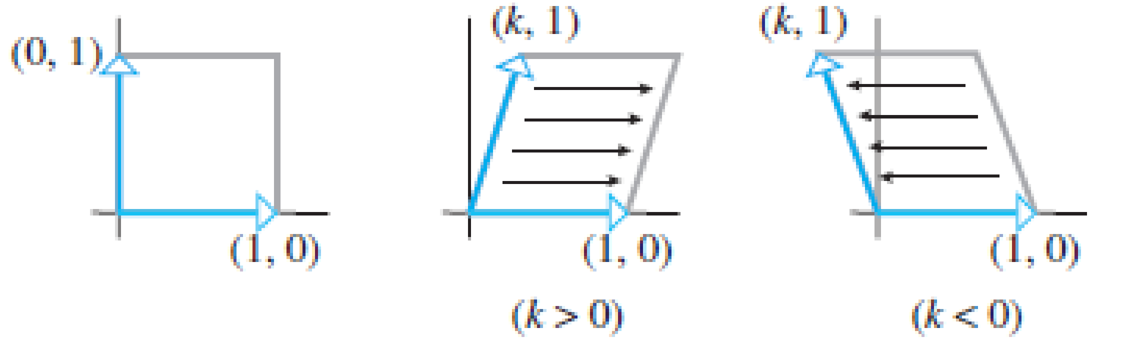
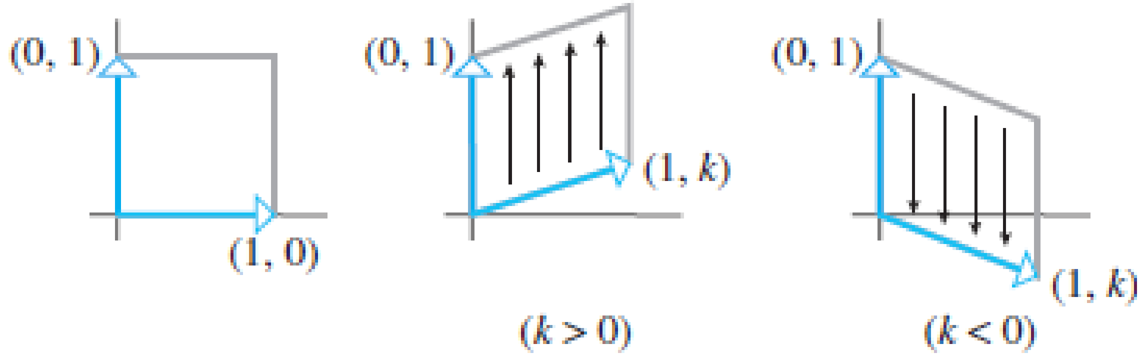
Table 9

Operator	Illustration $T(x, y) = (kx, y)$	Effect on the Standard Basis	Standard Matrix
Compression of R^2 in the x-direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion of R^2 in the x-direction with factor k $(k > 1)$			

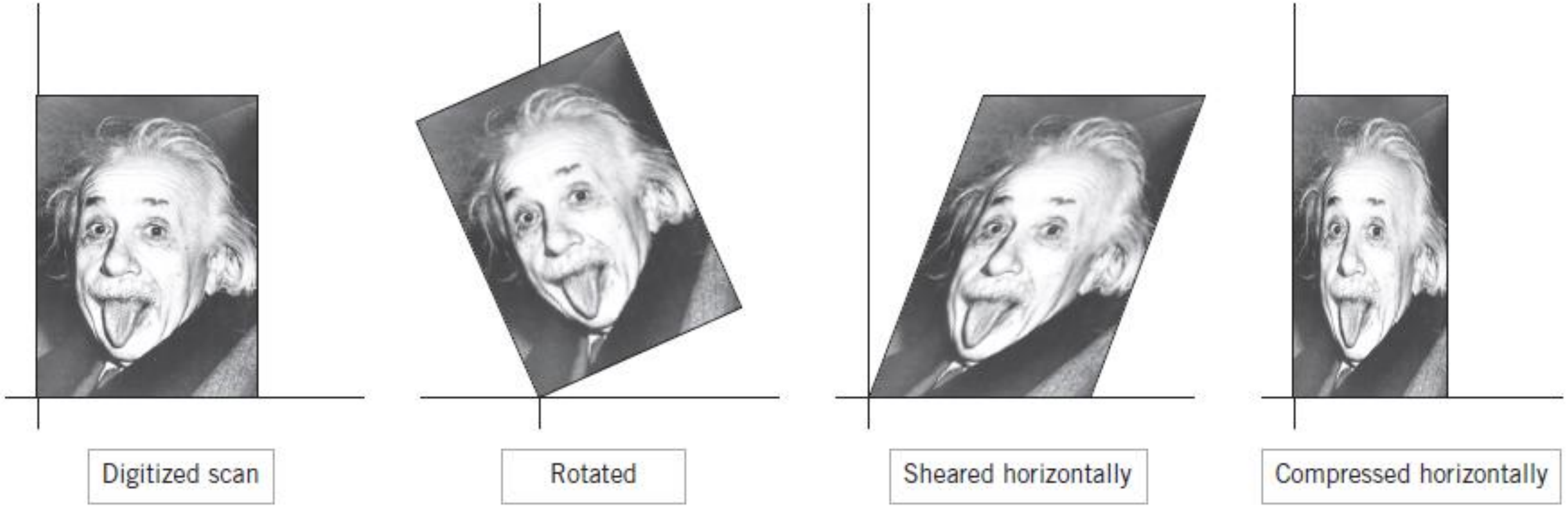
Operator	Illustration $T(x, y) = (x, ky)$	Effect on the Standard Basis	Standard Matrix
Compression of R^2 in the y-direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion of R^2 in the y-direction with factor k $(k > 1)$			

Shear

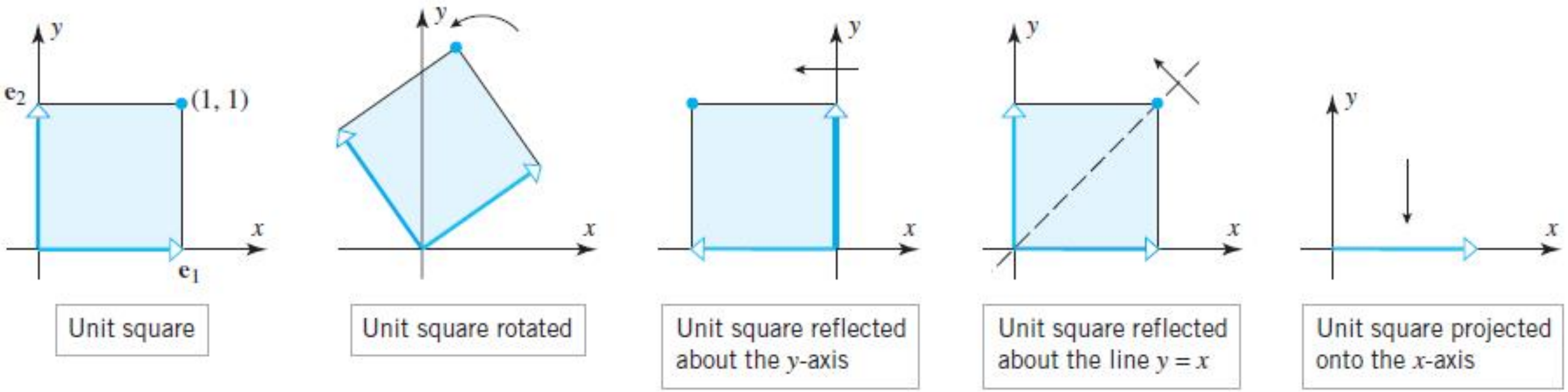
Table 10

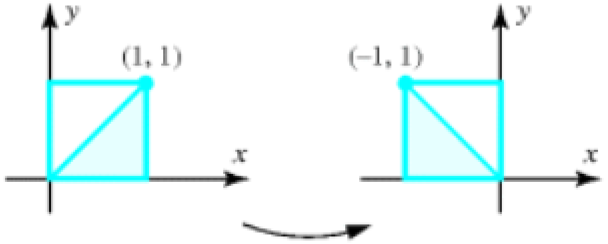
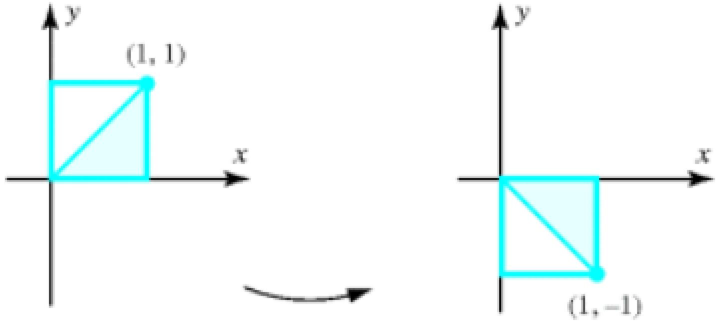
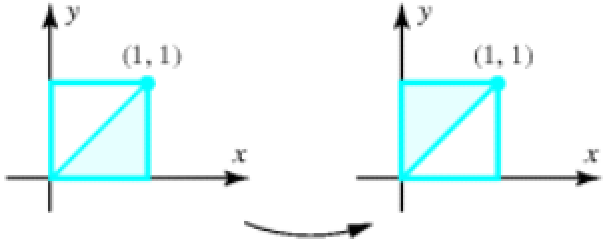
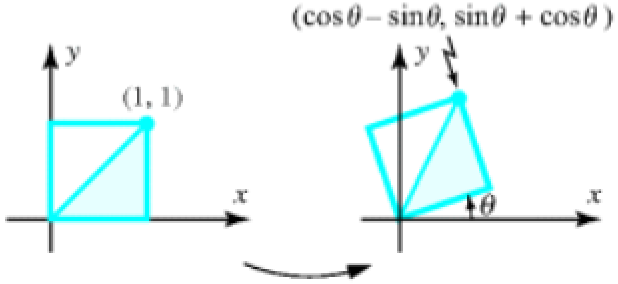
Operator	Effect on the Standard Basis	Standard Matrix
Shear of R^2 in the x-direction with factor k $T(x, y) = (x + ky, y)$		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear of R^2 in the y-direction with factor k $T(x, y) = (x, y + kx)$		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

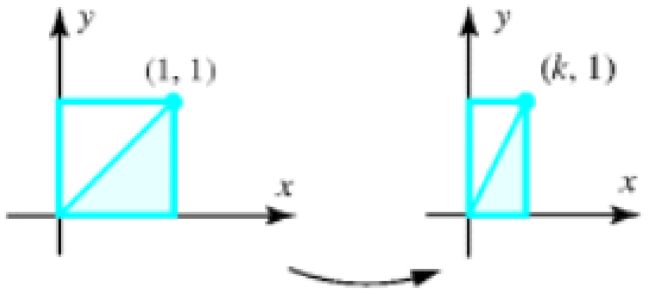
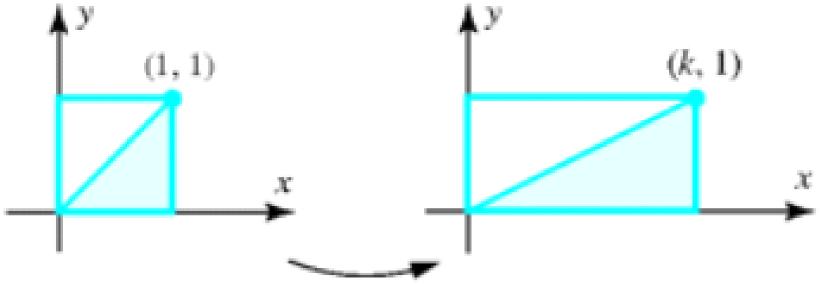
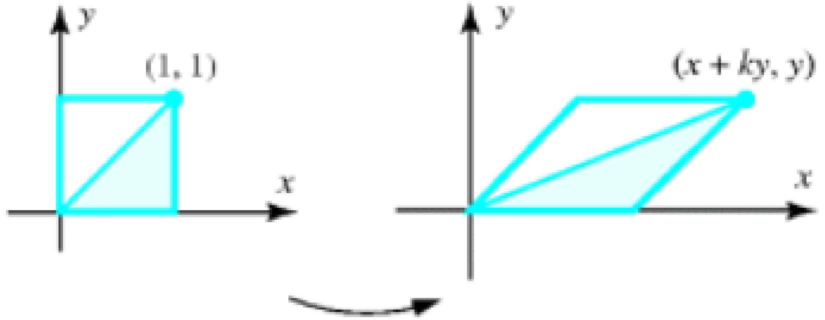
Geometri Operator Matriks di R^2



▲ **Figure 4.11.1**



Operator	Standard Matrix	Effect on the Unit Square
Reflection about the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	
Reflection about the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	
Reflection about the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	
Counterclockwise rotation through an angle θ	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	

Compression in the x-direction by a factor of k ($0 < k < 1$)	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	
Expansion in the x-direction by a factor of k ($k > 1$)	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	
Shear in the x-direction with factor $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	

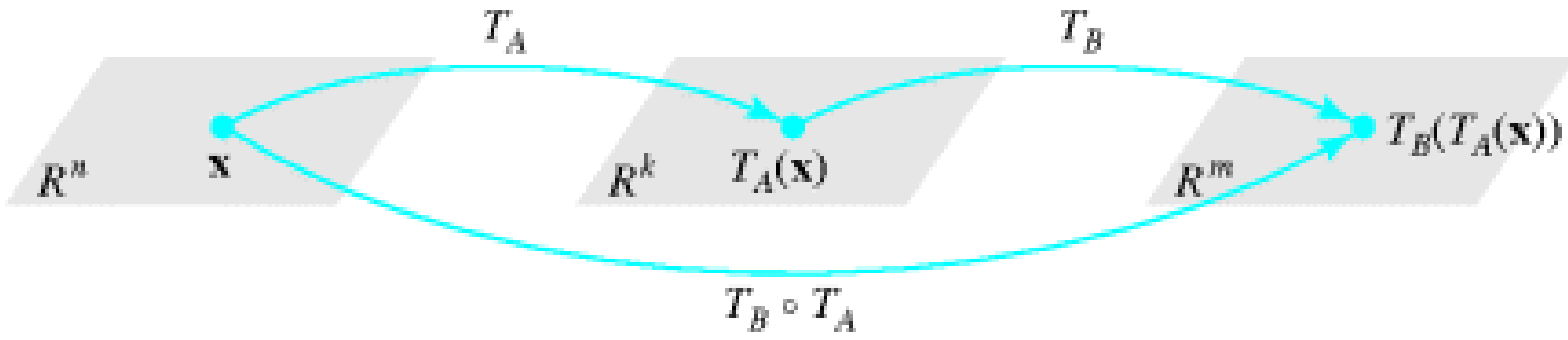
Komposisi Transformasi

- Misalkan T_A : $R^n \rightarrow R^k$ dan T_B : $R^k \rightarrow R^m$ maka jika sebuah vektor **x** ditransformasikan oleh T_A lalu bayangannya ditransformasikan lagi oleh T_B, maka hasilnya adalah transformasi dari R^n ke R^m yang dinamakan **komposisi T_B dengan T_A** dan dinyatakan dengan simbol:

$$T_B \text{ o } T_A$$

- Urutan pengerjaan adalah T_A dulu baru kemudian T_B, atau dinyatakan sebagai:

$$(T_B \text{ o } T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$$



- Komposisi transformasi ini sendiri adalah trransformasi matriks sebab:

$$(T_B \text{ o } T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) = B(T_A(\mathbf{x})) = (BA)\mathbf{x}$$

yang memperlihatkan bahwa ini adalah perjalian matriks BA.

Jadi,

$$T_B \text{ o } T_A = T_{BA}$$

- Perhatikan bahwa komposisi tranformasi tidak komutatif, jadi

$$T_B \text{ o } T_A \neq T_A \text{ o } T_B$$

Contoh 7: Carilah matriks transformasi dari R^2 ke R^2 jika mula-mula vektor **v** diregang (*shear*) dengan faktor sebesar 3 dalam arah-x kemudian hasilnya dicerminkan terhadap y = x.

Jawaban:

Matriks standard peregangan dalam arah x dengan faktor *k* = 3 adalah

$$A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Matriks standard pencerminan terhadap y = x adalah

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Jadi, matriks standard untuk peregangan lalu diikuti pencerminan adalah

$$A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

Jadi, T(**v**) = $\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$

- Contoh kombinasi transformasi lainnya: rotasi sejauh θ lalu diikuti dengan kompresi dalam arah x dengan factor ½.

Rotasi: $A_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Kompresi: $A_2 = \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix}$

Matriks standard rotasi lalu diikuti kompresi adalah

$$A_2A_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cos\theta & -\frac{1}{2}\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Secara umum, jika T_1, T_2, …, T_k adalah transformasi

$$T_1(\mathbf{x}) = A_1\mathbf{x}$$

$$T_2(\mathbf{x}) = A_2\mathbf{x}$$

$$\ldots$$

$$T_k(\mathbf{x}) = A_k\mathbf{x}$$

dari R^n ke R^n dan dilakukan secara berturut-turut (T_1, T_2, …, T_k), maka hasil yang sama dicapai dengan sebuah transformasi

$$T(\mathbf{x}) = A\mathbf{x}$$

yang dalam hal ini,

$$A = A_k A_{k-1} \ldots A_2 A_1$$

Latihan

1. Soal UAS 2017

Transformasi $T : \mathbb{R}^4 \to \mathbb{R}^3$ didefinisikan :

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7x_1 + 2x_2 - x_3 + x4 \\ x_2 + x_3 \\ -x_1 + 2x_3 \end{pmatrix}$$

a). Tentukan matriks transformasi T. (Perlihatkan cara perhitungan dengan menggunakan vektor basis satuan).

b). Dengan menggunakan jawab a), tentukan bayangan vektor (3,-1,4,5).

c). Jika hasil dari langkah b) diregang (shear) dalam arah x, tentukan bayangan akhirnya.

2. (Soal kuis 2017)

Tentukan bayangan dari vektor (-2,1,2) jika dirotasikan sebesar 30^o pada sumbu x.

3. (Soal UTS 2015)

Tinjaulah basis $S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$ untuk R^3 yang dalam hal ini $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (2, 5, 3)$, dan $\mathbf{v}_3 = (1, 0, 10)$. Carilah sebuah rumus untuk tansformasi lanjar T : $R^3 \rightarrow R^2$ sehingga $T(\mathbf{v}_1) = (1, 0)$, $T(\mathbf{v}_2) = (1, 0)$, dan $T(\mathbf{v}_3) = (0, 1)$, lalu hitunglah $T(1, 1, 1)$. **(20)**

Materi Pelengkap
(Opsional)

Aplikasi Transformasi Linier di dalam *Computer Graphics*

Oleh: Rinaldi Munir

Program Studi Informatika
Sekolah Teknik Elektro dan Informatika
ITB

Aplikasi Transformasi Linier di dalam *Computer Graphics*

- **Definisi**: Jika $T : V \rightarrow W$ adalah sebuah fungsi dari ruang vektor V ke ruang vektor W, maka T dinamakan transformasi linier jika

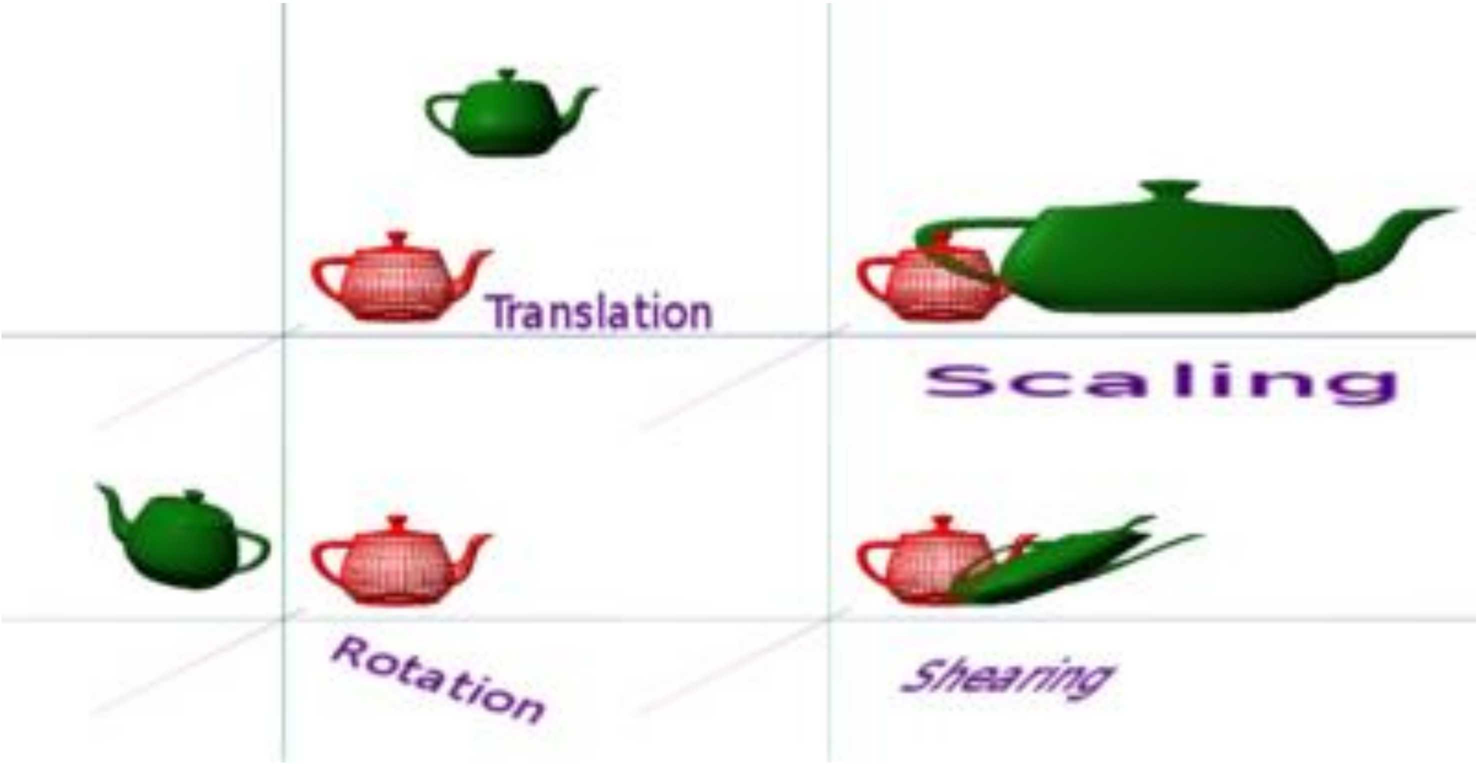
  (i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ untuk semua vektor **u** dan **v** di dalam **V**

  (ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ untuk semua vektor **u** di dalam V

- Transformasi linier $T : R^n \rightarrow R^m$ dapat dinyatakan sebagai sebuah perkalian matriks

$$T(\mathbf{x}) = A\mathbf{x}$$

Jenis-Jenis Tranformasi Linier 2D (T : $R^2 \rightarrow R^2$)



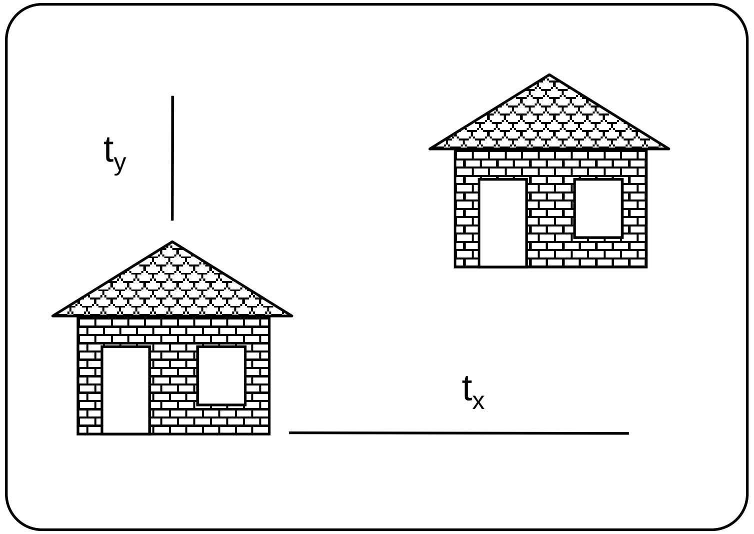
Sumber: http://mathforum.org/mathimages/imgUpload/thumb/Tranformations4.png/400px-Tranformations4.png

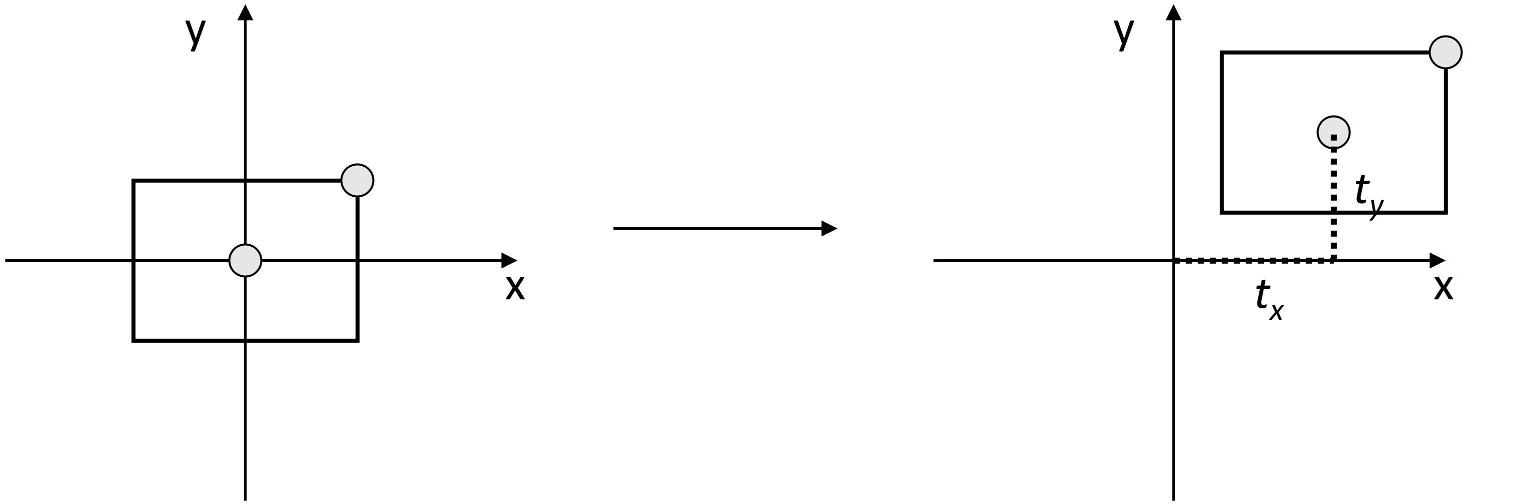
1. Translasi

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

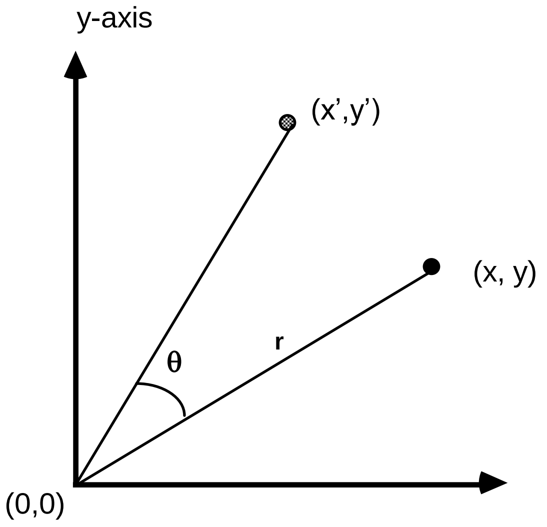
atau

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

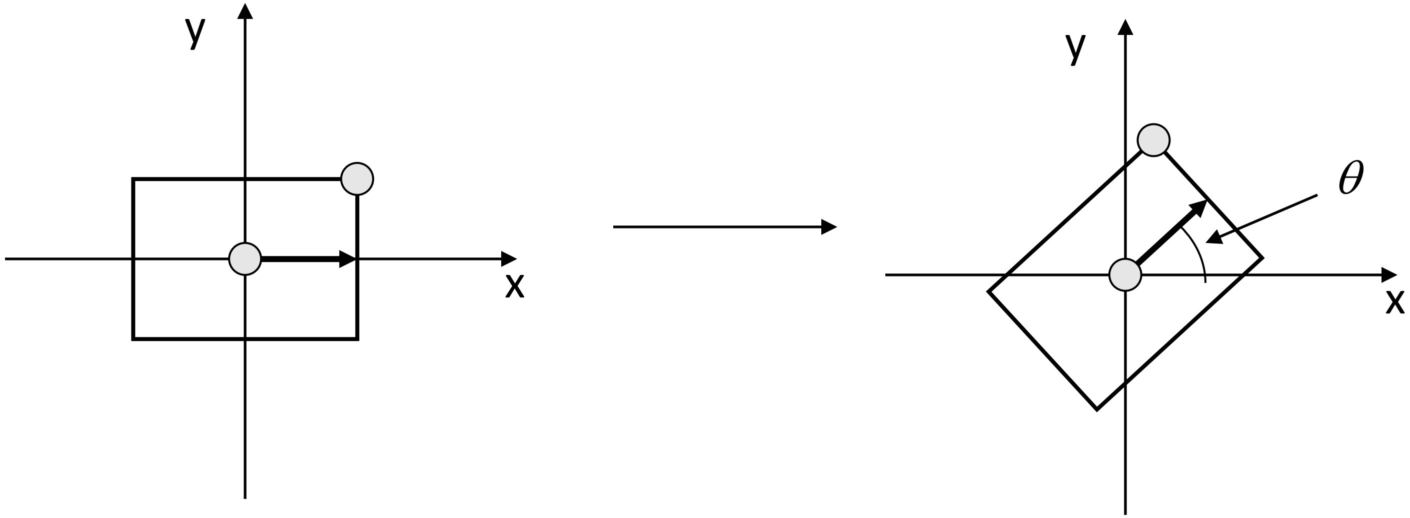




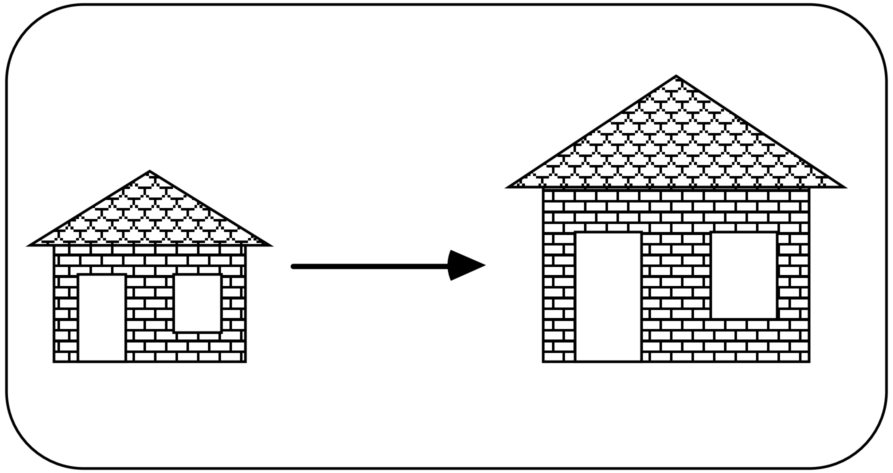
2. Rotasi

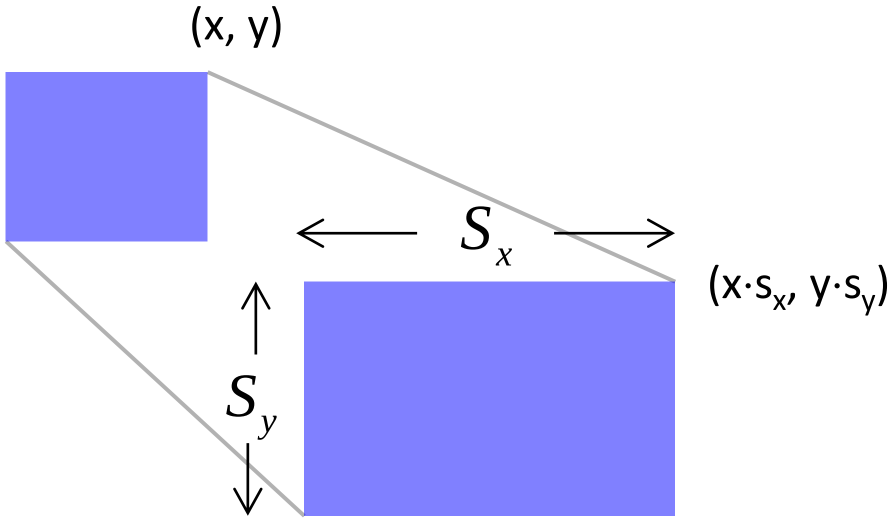


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



3. Penskalaan (*scaling*)





$$x' = x \cdot s_x, \quad y' = y \cdot s_y$$

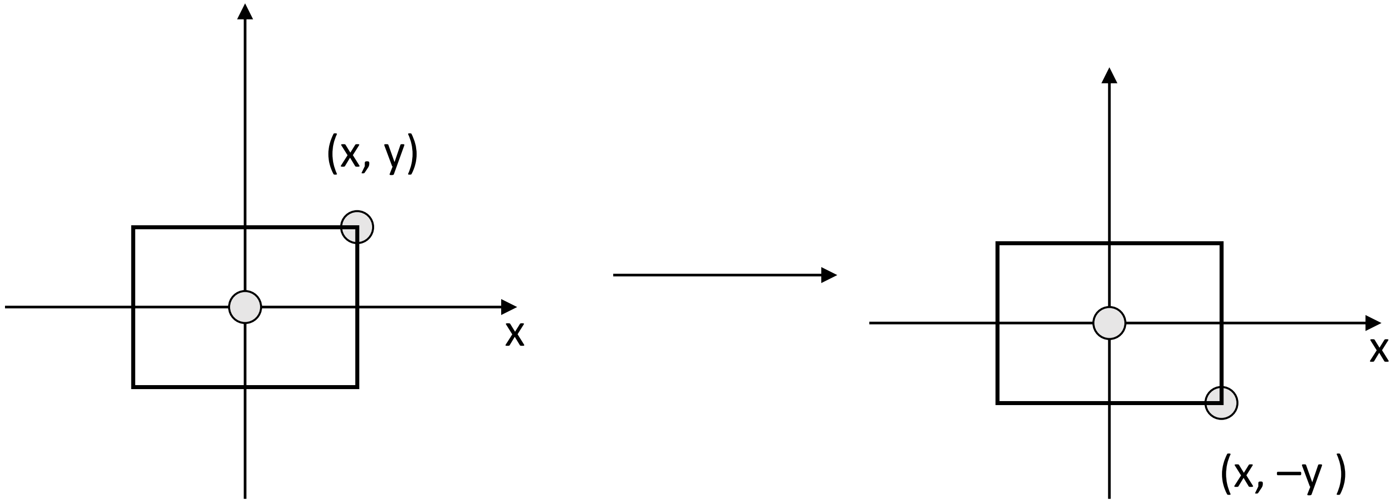
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

4. Pencerminan (*reflection*)

Pencerminan pada sumbu-X:

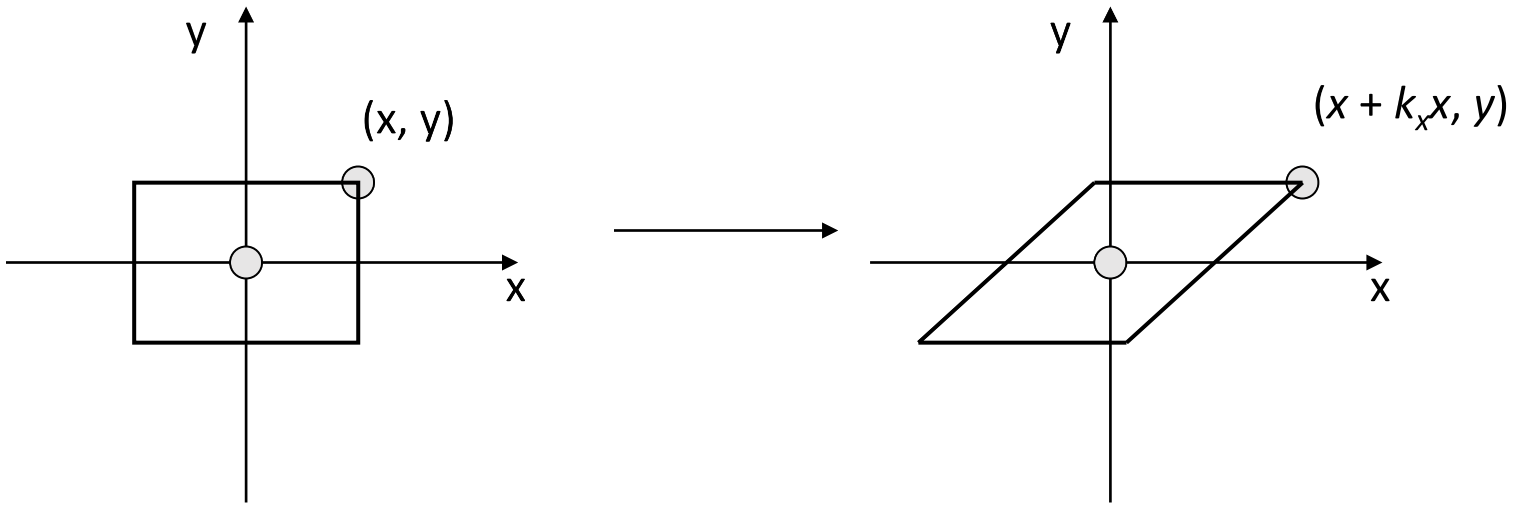
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



5. Peregangan (*shear*)

Peregangan sepanjang sumbu-X:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & k_x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Koordinat Homogen

- Di dalam grafika computer, sebuah gambar dapat dibangun dari dari sekumpulan bentuk terdefinisi (kotak, lingkaran, segitiga, dll).
- Tiap bentuk mungkin diskalakan, dirotasi, atau ditranslasi ke posisi gambar yang sebenarnya.
- Agar perhitungan koordinat akhir dapat langsung dihitung dari koordinat awal dengan efisien, maka diperlukan sebuah sistem koordinat yang homogen
- Pada koordinat homogen, setiap titik direpresentasikan dengan tiga angka:

$$(x, y) \rightarrow (x \cdot w, y \cdot w, w) \quad \text{dengan syarat } w \neq 0$$

Translasi 2D

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad \mathbf{P}' = \mathbf{T}\left(t_x, t_y\right) \cdot \mathbf{P}$$

Rotasi 2D

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad \mathbf{P}' = \mathbf{R}\left(\theta\right) \cdot \mathbf{P}$$

Penskalaan 2D

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad \mathbf{P}' = \mathbf{S}\left(S_x, S_y\right) \cdot \mathbf{P}$$

Transformasi *inverse*:

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{R}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{S}^{-1} = \begin{bmatrix} 1/S_x & 0 & 0 \\ 0 & 1/S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Komposisi tranformasi: $\mathbf{P}' = \mathbf{M}_2\left(\mathbf{M}_1 \cdot \mathbf{P}\right) = \left(\mathbf{M}_2 \cdot \mathbf{M}_1\right) \cdot \mathbf{P} = \mathbf{M} \cdot \mathbf{P}$

$$\mathbf{P}' = \mathbf{T}\left(t_{2x}, t_{2y}\right)\left\{\mathbf{T}\left(t_{1x}, t_{1y}\right) \cdot \mathbf{P}\right\} = \left\{\mathbf{T}\left(t_{2x}, t_{2y}\right) \cdot \mathbf{T}\left(t_{1x}, t_{1y}\right)\right\} \cdot \mathbf{P}$$

Komposisi translasi:

$$\begin{bmatrix} 1 & 0 & t_{2x} \\ 0 & 1 & t_{2y} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{1x} \\ 0 & 1 & t_{1y} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{1x} + t_{2x} \\ 0 & 1 & t_{1y} + t_{2y} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}\left(t_{2x}, t_{2y}\right) \cdot \mathbf{T}\left(t_{1x}, t_{1y}\right) = \mathbf{T}\left(t_{1x} + t_{2x}, t_{1y} + t_{2y}\right)$$

Komposisi rotasi:

$$\mathbf{P}' = \mathbf{R}\left(\theta_2\right)\left\{\mathbf{R}\left(\theta_1\right) \cdot \mathbf{P}\right\} = \left\{\mathbf{R}\left(\theta_2\right) \cdot \mathbf{R}\left(\theta_1\right)\right\} \cdot \mathbf{P}$$

$$\mathbf{R}\left(\theta_2\right) \cdot \mathbf{R}\left(\theta_1\right) = \mathbf{R}\left(\theta_1 + \theta_2\right)$$

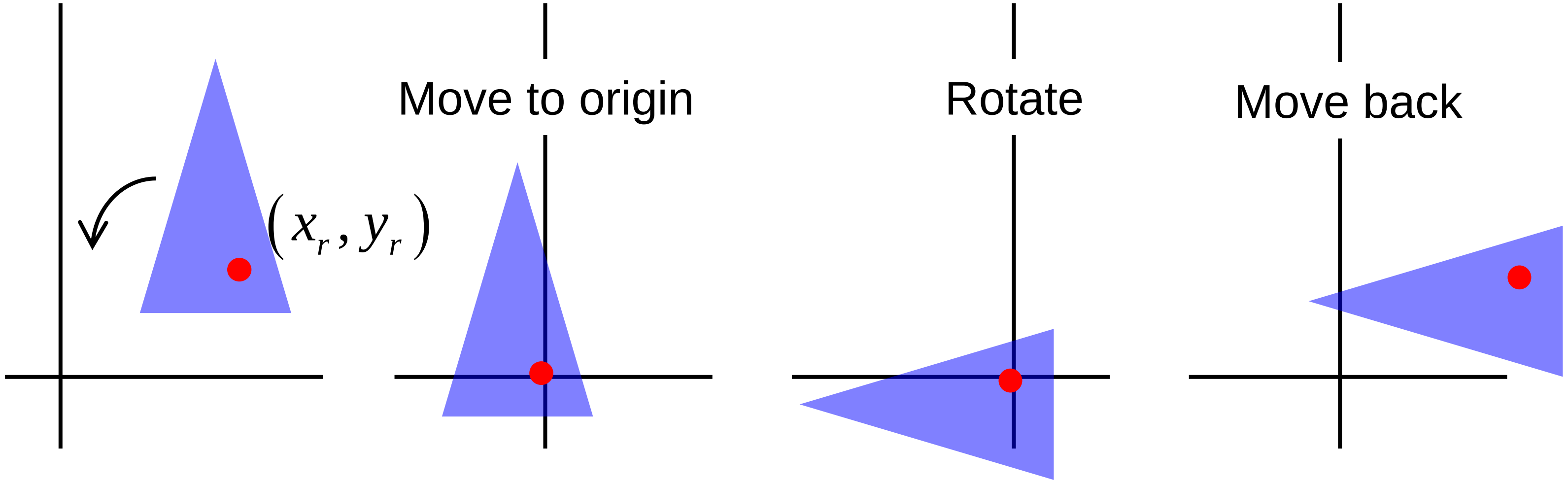
$$\mathbf{P}' = \mathbf{R}\left(\theta_1 + \theta_2\right) \cdot \mathbf{P}$$

$$\begin{bmatrix} S_{2x} & 0 & 0 \\ 0 & S_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_{1x} & 0 & 0 \\ 0 & S_{1y} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_{1x} \cdot S_{2x} & 0 & 0 \\ 0 & S_{1y} \cdot S_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Komposisi penskalaan:

$$\mathbf{S}\left(S_{2x}, S_{2y}\right) \cdot \mathbf{S}\left(S_{1x}, S_{1y}\right) = \mathbf{S}\left(S_{1x} \cdot S_{2x}, S_{1y} \cdot S_{2y}\right)$$

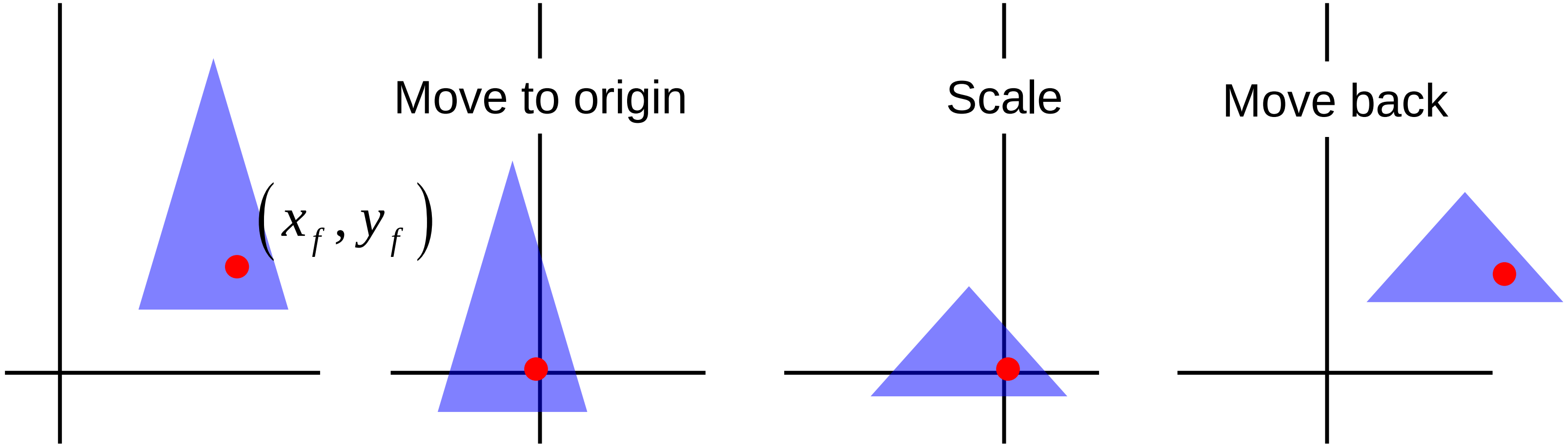
General 2D Rotation



$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta)+y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta)-x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

General 2D Scaling

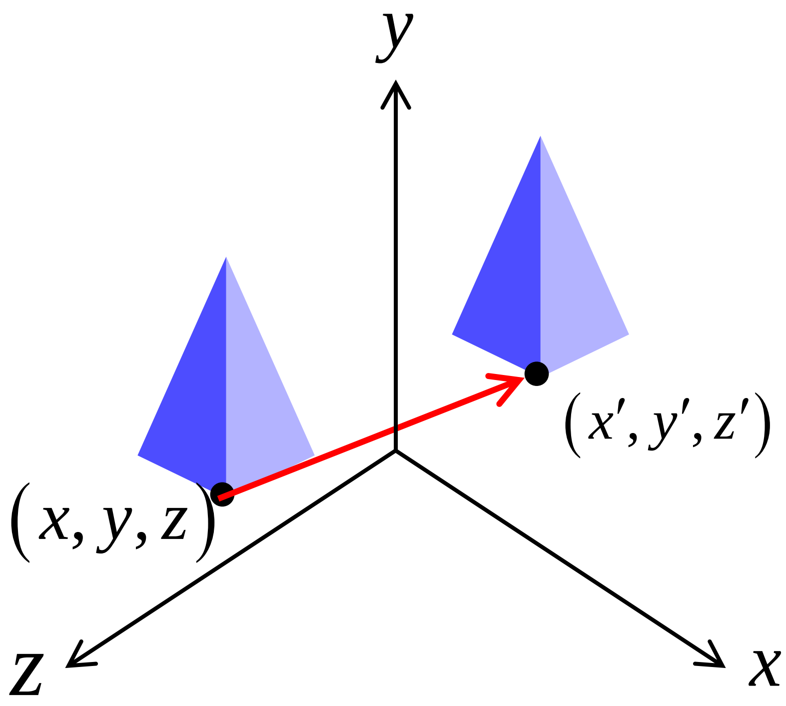


$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & x_f(1-S_x) \\ 0 & S_y & y_f(1-S_y) \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation 3D

Mirip dengan transformasi 2D. Menggunakan matriks 4x4

Translation



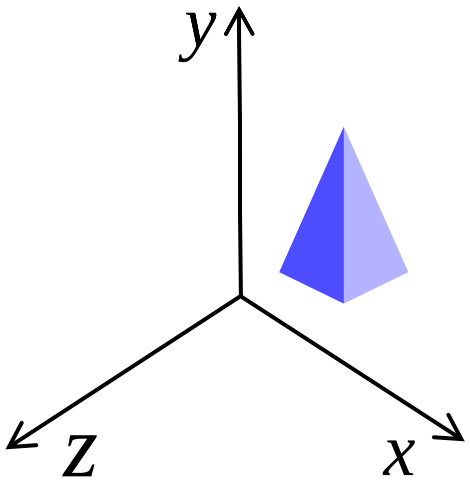
$$x' = x + t_x$$

$$y' = y + t_y$$

$$z' = z + t_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

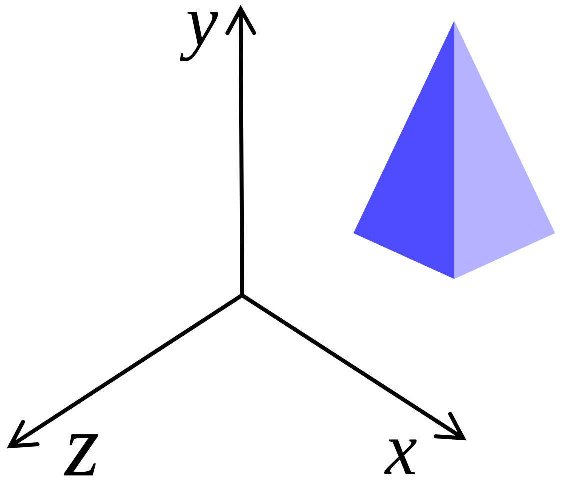
Penskalaan 3D



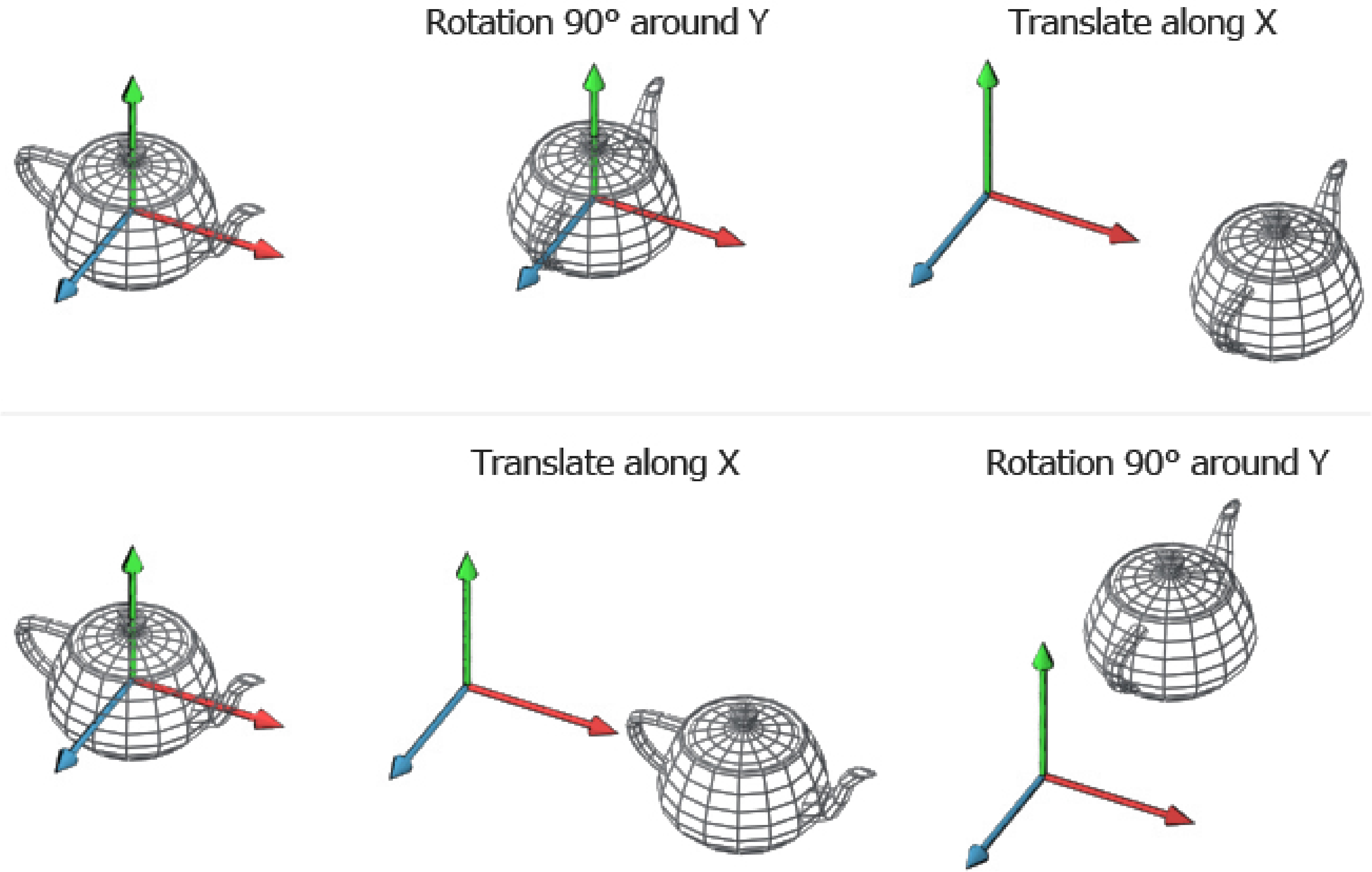
$$x' = x \cdot S_x$$

$$y' = y \cdot S_y$$

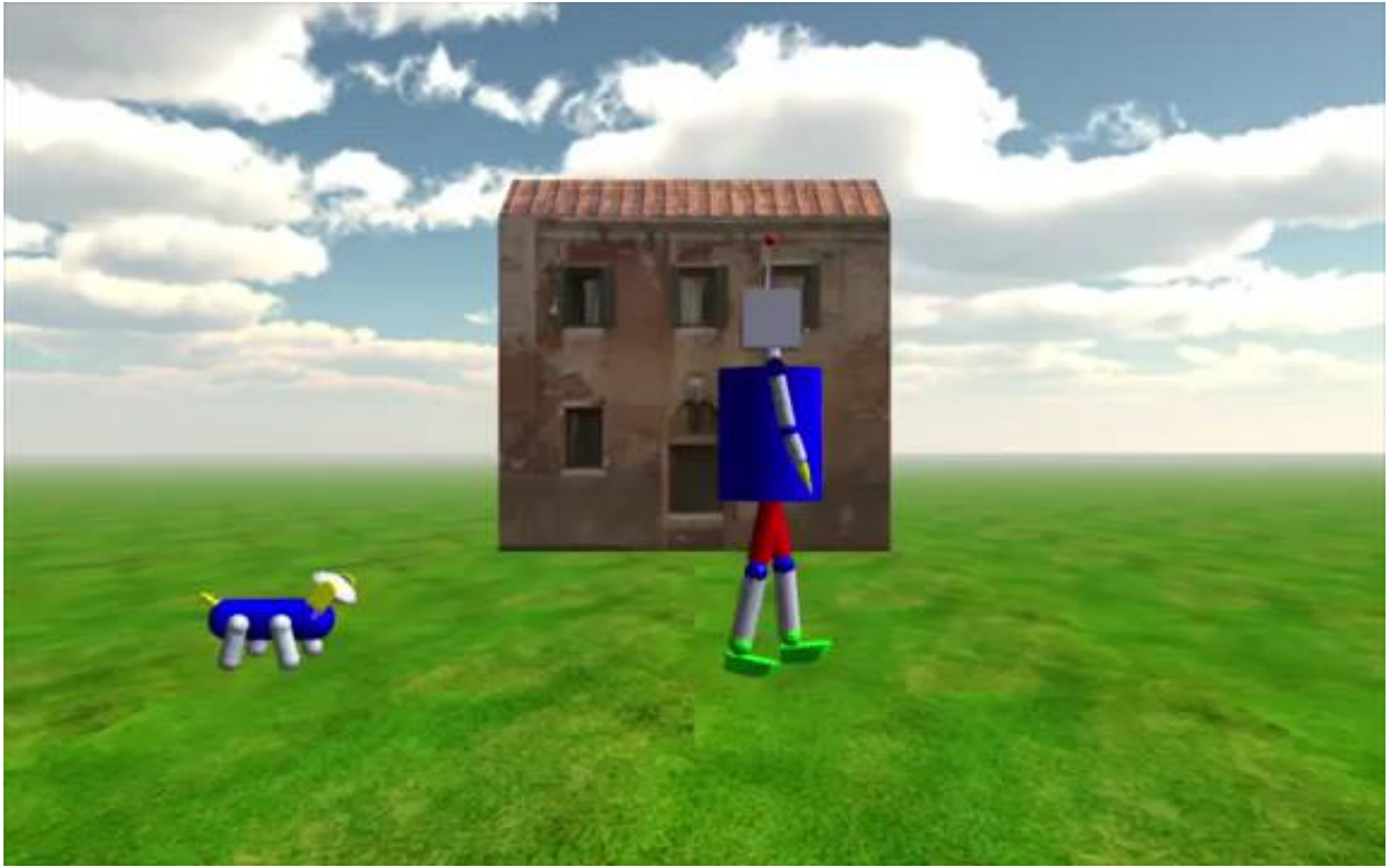
$$z' = x \cdot S_z$$



$$\mathbf{P}' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{S} \cdot \mathbf{P}$$



Sumber: http://www.codinglabs.net/public/contents/article_world_view_projection_matrix/images/order_dependency.png



Referensi

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