

Seri bahan kuliah Algeo #27

Aljabar Geometri (Bagian 2)

Bahan kuliah IF2123 Aljabar Linier dan Geometri

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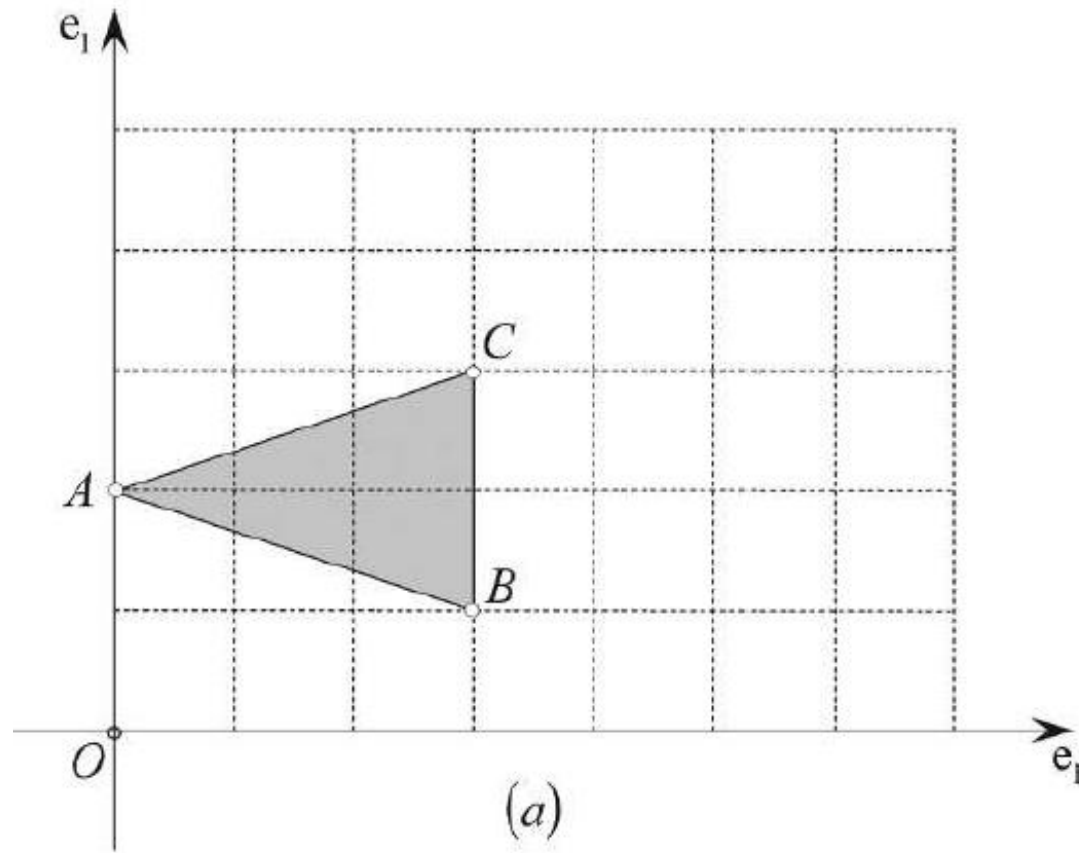
Sumber:

John Vince, *Geometric Algebra for Computer Graphics*. Springer. 2007

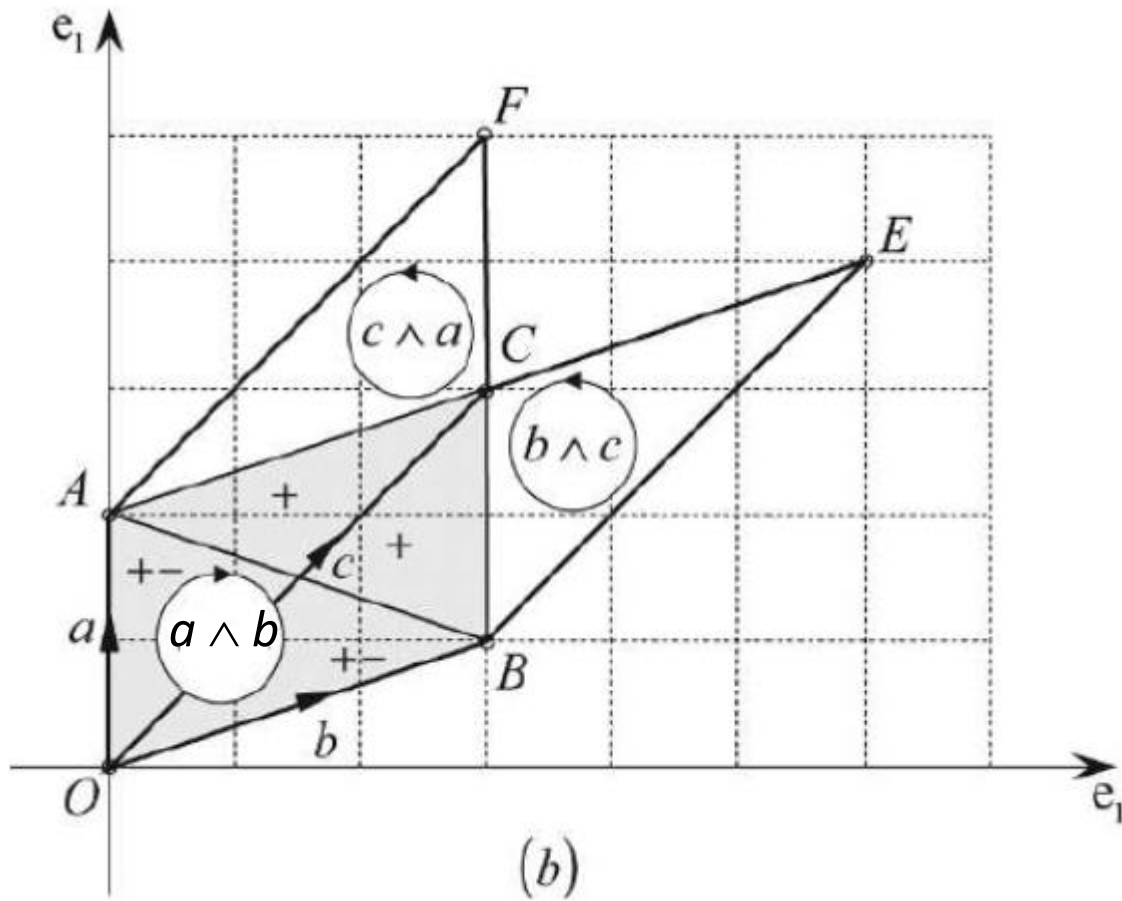
Aplikasi Aljabar Geometri

1. Menghitung luas segitiga
2. Menghitung volume *parallelepiped*
3. Menghitung perpotongan dua garis

1. Menghitung Luas Segitiga



Berapa luas segitiga ABC?



Misalkan:

$$a = x_A e_1 + y_A e_2$$

$$b = x_B e_1 + y_B e_2$$

$$c = x_C e_1 + y_C e_2$$

$a \wedge b$: menghitung luas OBCA

$$\frac{1}{2} (a \wedge b) = \text{luas OBA}$$

$b \wedge c$: menghitung luas OBEC

$$\frac{1}{2} (b \wedge c) = \text{luas OBC}$$

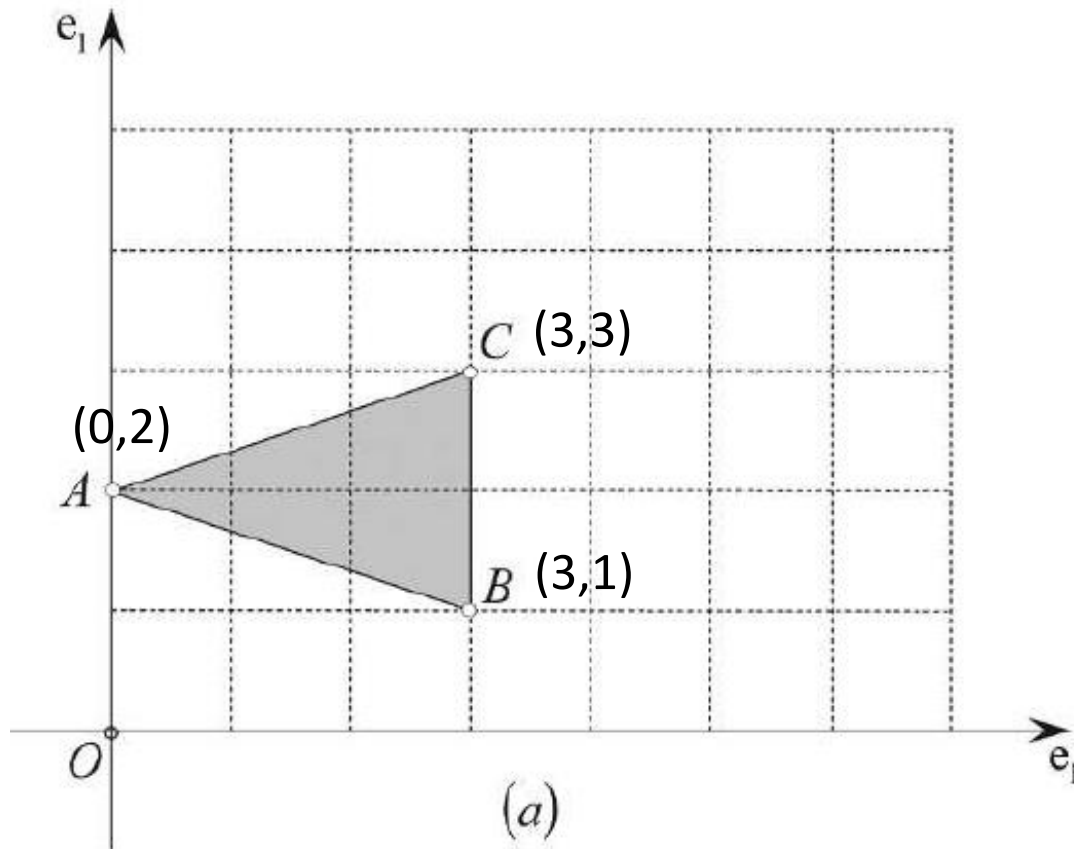
$c \wedge a$: menghitung luas OCFA

$$\frac{1}{2} (c \wedge a) = \text{luas OCA}$$

$$\text{Luas } \Delta ABC = \frac{1}{2} [(a \wedge b) + (b \wedge c) + (c \wedge a)]$$

$$= \frac{1}{2} (x_A y_B - y_A x_B + x_B y_C - y_B x_C + x_C y_A - y_C x_A) = \frac{1}{2} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}$$

Contoh 1: Hitunglah luas segitiga ABC berikut dengan menggunakan *outer product*.

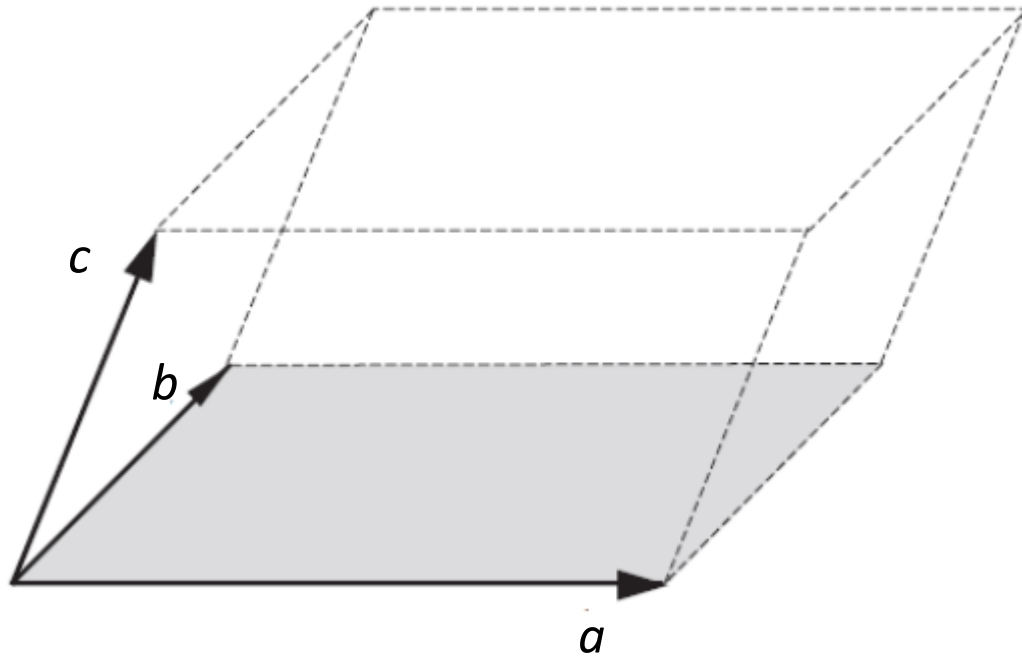


$$\begin{aligned}\text{Luas segitiga ABC} &= \frac{1}{2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix} \\ &= \frac{1}{2}(9 + 6 - 6 - 3) = +3\end{aligned}$$

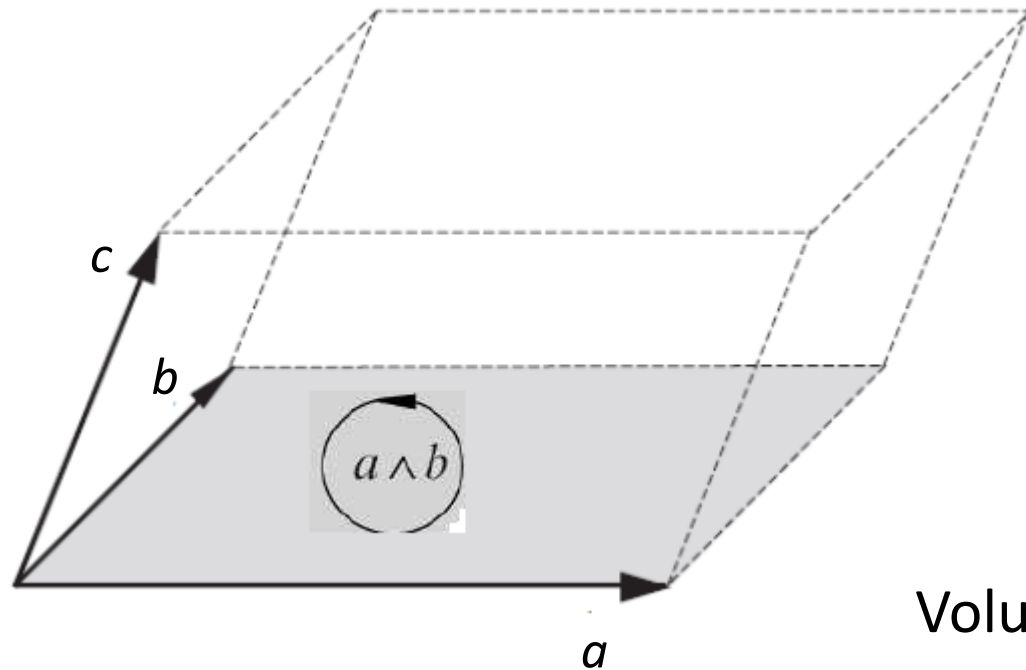
Perhatikan, jika urutannya dibalik maka hasilnya negatif:

$$\begin{aligned}\text{Luas segitiga ABC} &= \frac{1}{2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2}(3 + 6 - 6 - 9) = -3\end{aligned}$$

2. Menghitung volume *parallelepiped*



Berapa volume *parallelepiped* ini?



Misalkan:

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3$$

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3$$

$$c = c_1 e_1 + c_2 e_2 + c_3 e_3$$

Volume *parallelepiped* adalah:

$$(a \wedge b) \wedge c = (b \wedge c) \wedge a = (c \wedge a) \wedge b$$

Bentuk $(a \wedge b) \wedge c$ dinamakan *trivector*

$$\begin{aligned}
a \wedge b \wedge c &= (a_1e_1 + a_2e_2 + a_3e_3) \wedge (b_1e_1 + b_2e_2 + b_3e_3) \wedge (c_1e_1 + c_2e_2 + c_3e_3) \\
&= \begin{pmatrix} a_1b_1e_1 \wedge e_1 + a_1b_2e_1 \wedge e_2 + a_1b_3e_1 \wedge e_3 + \\ a_2b_1e_2 \wedge e_1 + a_2b_2e_2 \wedge e_2 + a_2b_3e_2 \wedge e_3 + \\ a_3b_1e_3 \wedge e_1 + a_3b_2e_3 \wedge e_2 + a_3b_3e_3 \wedge e_3 \end{pmatrix} \wedge (c_1e_1 + c_2e_2 + c_3e_3) \\
&= \begin{pmatrix} a_1b_2e_1 \wedge e_2 - a_1b_3e_3 \wedge e_1 - a_2b_1e_1 \wedge e_2 + \\ a_2b_3e_2 \wedge e_3 + a_3b_1e_3 \wedge e_1 - a_3b_2e_2 \wedge e_3 \end{pmatrix} \wedge (c_1e_1 + c_2e_2 + c_3e_3) \\
a \wedge b \wedge c &= \begin{pmatrix} (a_1b_2 - a_2b_1)e_1 \wedge e_2 + (a_2b_3 - a_3b_2)e_2 \wedge e_3 \\ +(a_3b_1 - a_1b_3)e_3 \wedge e_1 \end{pmatrix} \wedge (c_1e_1 + c_2e_2 + c_3e_3)
\end{aligned}$$

Pada operasi *wedge product* di atas akan muncul bentuk:

$e_1 \wedge e_2 \wedge e_3 \rightarrow$ menyatakan volume satuan, dibangun oleh bivektor satuan $e_1 \wedge e_2$ dan vektor e_3

$e_1 \wedge e_2 \wedge e_1 \rightarrow$ tidak menyatakan volume

$e_1 \wedge e_2 \wedge e_2, \rightarrow$ tidak menyatakan volume

dst

Jadi,

$$e_1 \wedge e_1 \wedge e_1 = 0$$

$$e_2 \wedge e_2 \wedge e_2 = 0$$

$$e_3 \wedge e_3 \wedge e_3 = 0,$$

dst

Sehingga

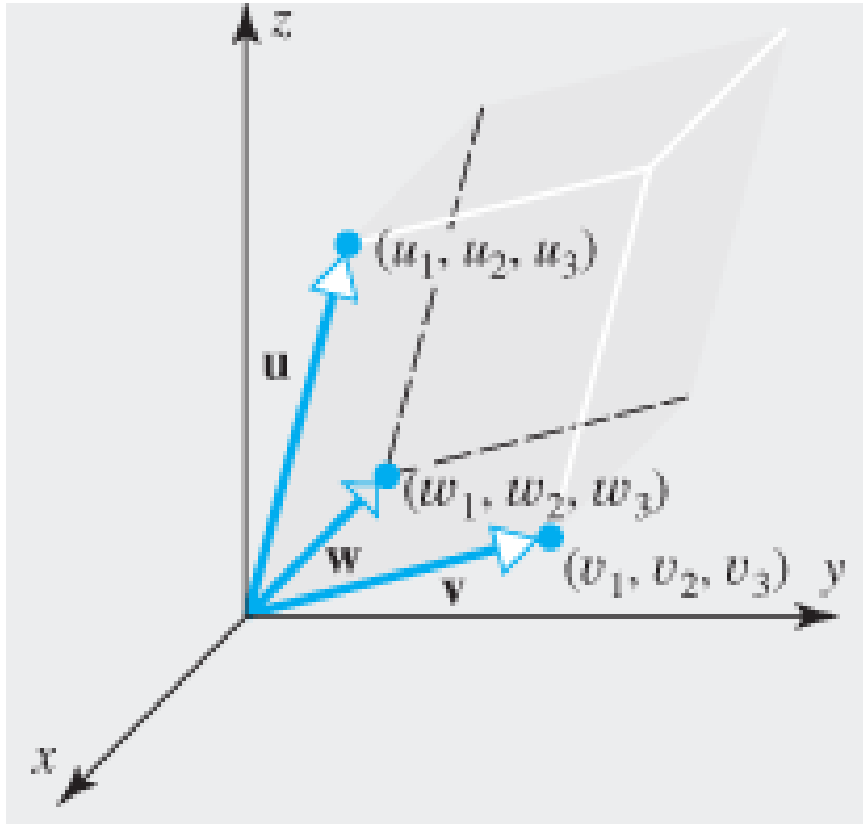
$$\begin{aligned} a \wedge b \wedge c &= (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 \wedge e_3 + (a_2 b_3 - a_3 b_2) e_1 \wedge e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) c_2 e_1 \wedge e_2 \wedge e_3 \\ &= ((a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3) e_1 \wedge e_2 \wedge e_3 \end{aligned}$$

Jadi, volume *parallelepiped* adalah:

$$a \wedge b \wedge c = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} e_1 \wedge e_2 \wedge e_3$$

Trivector $e_1 \wedge e_2 \wedge e_3$ menentukan arah volume (*signed volume*)

- Perhatikan bahwa rumus volume ini tidak bertentangan dengan rumus volume yang sudah dipelajari pada aljabar vektor:



Tinjau tiga vektor:

$$\mathbf{u} = (u_1, u_2, u_3)$$

$$\mathbf{v} = (v_1, v_2, v_3)$$

$$\mathbf{w} = (w_1, w_2, w_3)$$

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \right) \\ &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} u_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} u_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} u_3 \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{aligned}$$

Nilai mutlak dari determinan, atau $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$, menyatakan volume *parallelepiped*

Contoh 2 (Soal UAS 2019): Diketahui tiga buah vektor:

$$a = 2e_1 + 2e_2 + e_3$$

$$b = 3e_1 + 2e_2 - 2e_3$$

$$c = e_1 + 2e_2 - e_3$$

Hitunglah volume *parallelepiped* yang dibentuk oleh vektor a , b , dan c

Jawaban:

Volume *parallelepiped* adalah:

$$\begin{aligned} a \wedge b \wedge c &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} e_1 \wedge e_2 \wedge e_3 = \begin{vmatrix} 2 & 3 & 1 \\ 2 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix} e_1 \wedge e_2 \wedge e_3 \\ &= 10 e_1 \wedge e_2 \wedge e_3 \end{aligned}$$

Magnitude volume *parallelepiped* = $\|10 e_1 \wedge e_2 \wedge e_3\| = 10$

Perhatikan, jika urutannya dibalik maka hasilnya negatif:

$$\text{volume parallelepiped} = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ -2 & 1 & -1 \end{vmatrix} = -10 \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

yang menyatakan volume berarah atau bertanda,

namun *magnitude* volumenya tetap $\| -10 \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \| = 10$

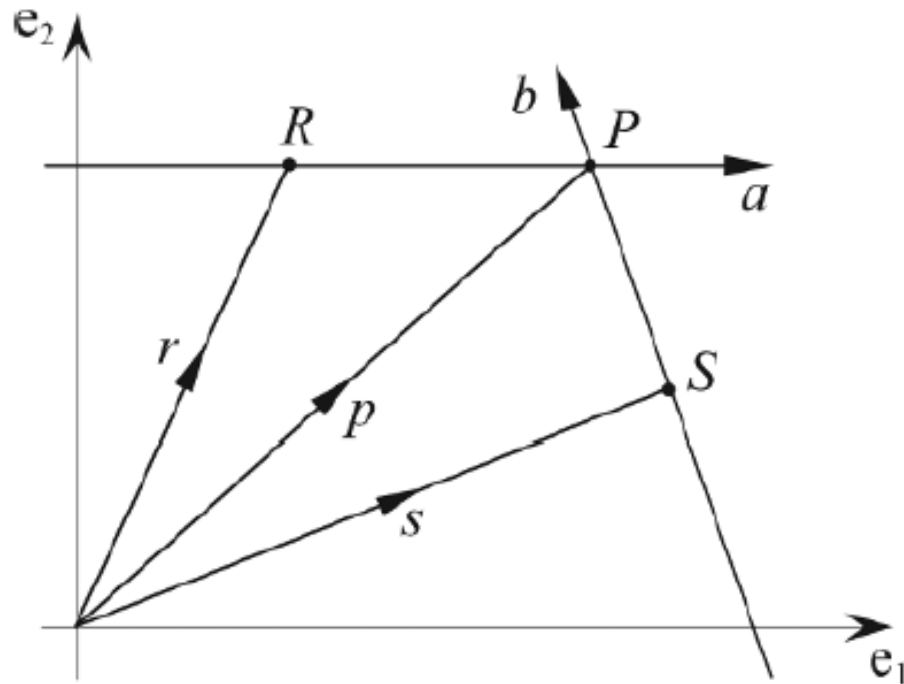
Latihan (Soal UAS 2018)

Diketahui tiga buah vektor, hitunglah

$$a = 3e_1 + 2e_2 - 2e_3; \quad b = e_1 - 2e_2 + 3e_3; \quad c = 2e_1 + e_2$$

1. Luas parallelogram yang dibentuk oleh vektor a dan b
2. Volume parallelepiped yang dibentuk oleh ketiga vektor tersebut.

3. Menghitung perpotongan dua buah garis



Garis a melalui titik R ,
Garis b melalui titik S
Keduanya berpotongan pada titik P
Tentukan titik P .

Misalkan: $a = a_1e_1 + a_2e_2$ dan $b = b_1e_1 + b_2e_2$

dan $p = \alpha a + \beta b$

$$x_p = \alpha x_a + \beta x_b$$

Koordinat P adalah:

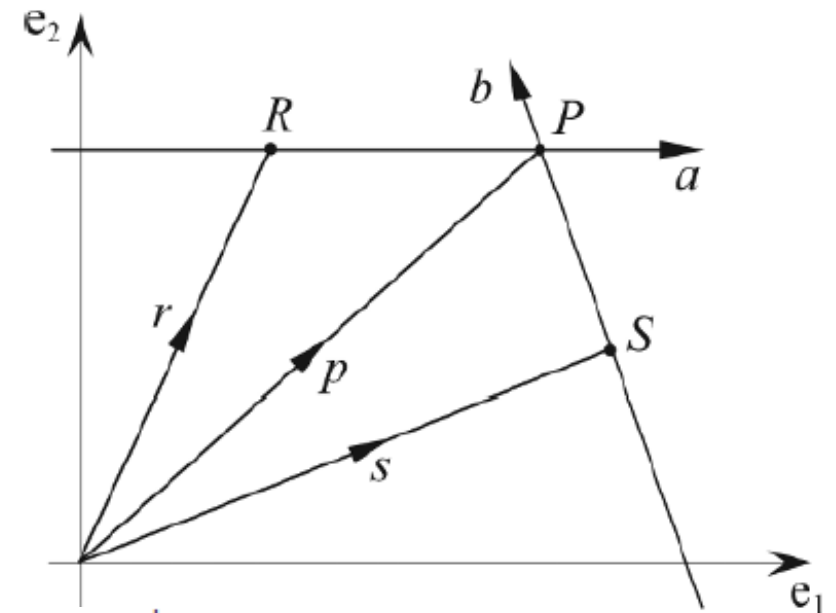
$$y_p = \alpha y_a + \beta y_b.$$

Nilai α dan β adalah:

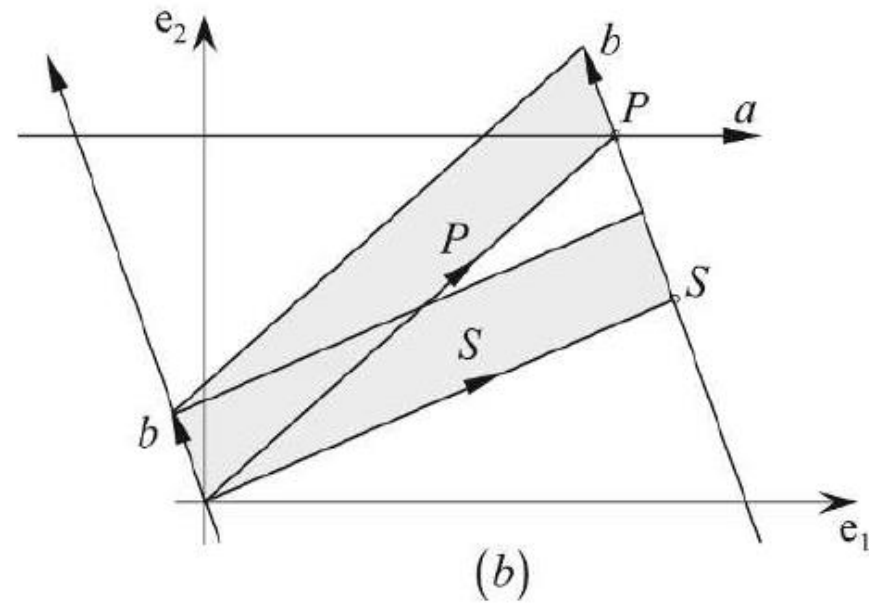
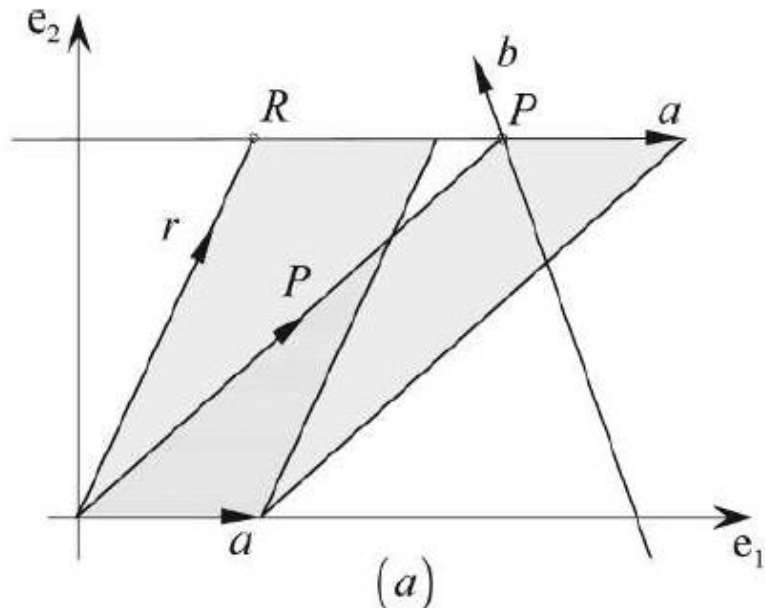
$$\alpha = \frac{x_p y_b - x_b y_p}{x_a y_b - x_b y_a} = \frac{\begin{vmatrix} x_p & y_p \\ x_b & y_b \end{vmatrix}}{\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}} \quad \beta = \frac{x_p y_a - x_a y_p}{x_b y_a - x_a y_b} = \frac{\begin{vmatrix} x_p & y_p \\ x_a & y_a \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}}$$

Sehingga,

$$p = \frac{\begin{vmatrix} x_p & y_p \\ x_b & y_b \end{vmatrix}}{\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}} a + \frac{\begin{vmatrix} x_p & y_p \\ x_a & y_a \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}} b \quad \longrightarrow \quad p = \frac{p \wedge b}{a \wedge b} a + \frac{p \wedge a}{b \wedge a} b$$



- Perhatikan dari dua gambar di bawah ini, $p \wedge a$ identik dengan $r \wedge a$ (Gambar a) dan $p \wedge b$ identik dengan $s \wedge b$ (Gambar b)



- Sehingga, $p = \frac{p \wedge b}{a \wedge b} a + \frac{p \wedge a}{b \wedge a} b$



$$p = \frac{s \wedge b}{a \wedge b} a + \frac{r \wedge a}{b \wedge a} b$$

Contoh 3: Misalkan $a = 2e_1 - e_2$ dan $b = 2e_1 - 2e_2$. R dan S adalah titik pada masing-masing a dan b , yaitu $R(0, 1)$ dan $S(0, 2)$. Tentukan titik potong vektor a dan b .

Jawaban:

$$r = 0e_1 + e_2 = e_2$$

$$s = 0e_1 + 2e_2 = 2e_2$$

$$p = \frac{s \wedge b}{a \wedge b} a + \frac{r \wedge a}{b \wedge a} b$$

$$\begin{aligned} p &= \frac{(2e_2) \wedge (2e_1 - 2e_2)}{(2e_1 - e_2) \wedge (2e_1 - 2e_2)} (2e_1 - e_2) + \frac{e_2 \wedge (2e_1 - e_2)}{(2e_1 - 2e_2) \wedge (2e_1 - e_2)} (2e_1 - 2e_2) \\ &= \frac{-4(e_1 \wedge e_2)}{-4(e_1 \wedge e_2) + 2(e_1 \wedge e_2)} (2e_1 - e_2) + \frac{-2(e_1 \wedge e_2)}{-2(e_1 \wedge e_2) + 4(e_1 \wedge e_2)} (2e_1 - 2e_2) \\ &= 2(2e_1 - e_2) - (2e_1 - 2e_2) = 2e_1. \end{aligned}$$

Jadi, titik potong kedua vektor adalah $P(2, 0)$

TAMAT