Seri bahan kuliah Algeo #25

Aljabar Quaternion (Bagian 2)

Bahan kuliah IF2123 Aljabar Linier dan Geometri

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STEI-ITB

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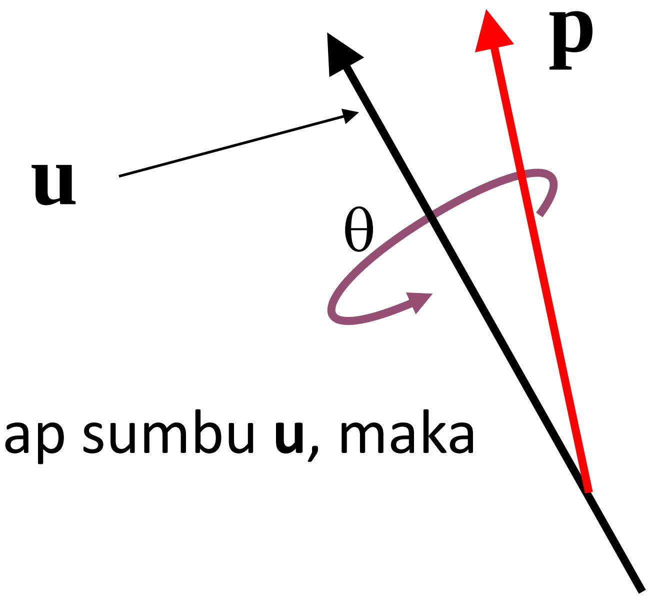
Sumber:

John Vince, *Geometric Algebra for Computer Graphics*. Springer. 2007

Rotasi Vektor dengan Quaternion



- Misalkan **p** adalah sebuah vektor di R^3
- Vektor **p** diputar sejauh θ berlawanan arah jarum jam terhadap sumbu **u**, maka bayangannya adalah **p**', yang dihitung dengan persamaan:

$$\boxed{\mathbf{p}' = q\mathbf{p}q^{-1}}$$

yang dalam hal ini,

$$\mathrm{p} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$$

$$p = 0 + ix + jy + kz$$

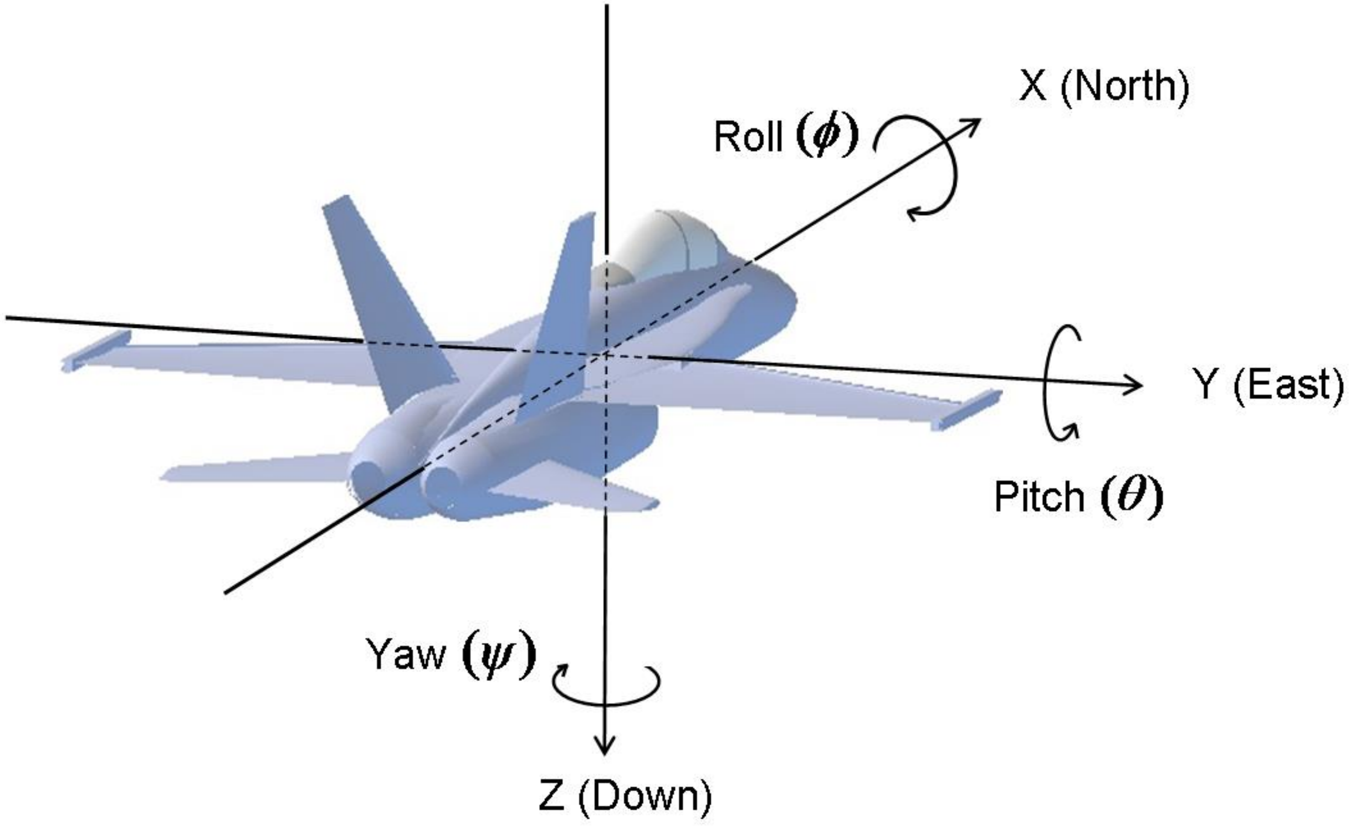
$$q = \cos(\theta/2) + \sin(\theta/2)\hat{\mathrm{u}}$$

$$q^{-1} = \cos(\theta/2) - \sin(\theta/2)\hat{\mathrm{u}}$$

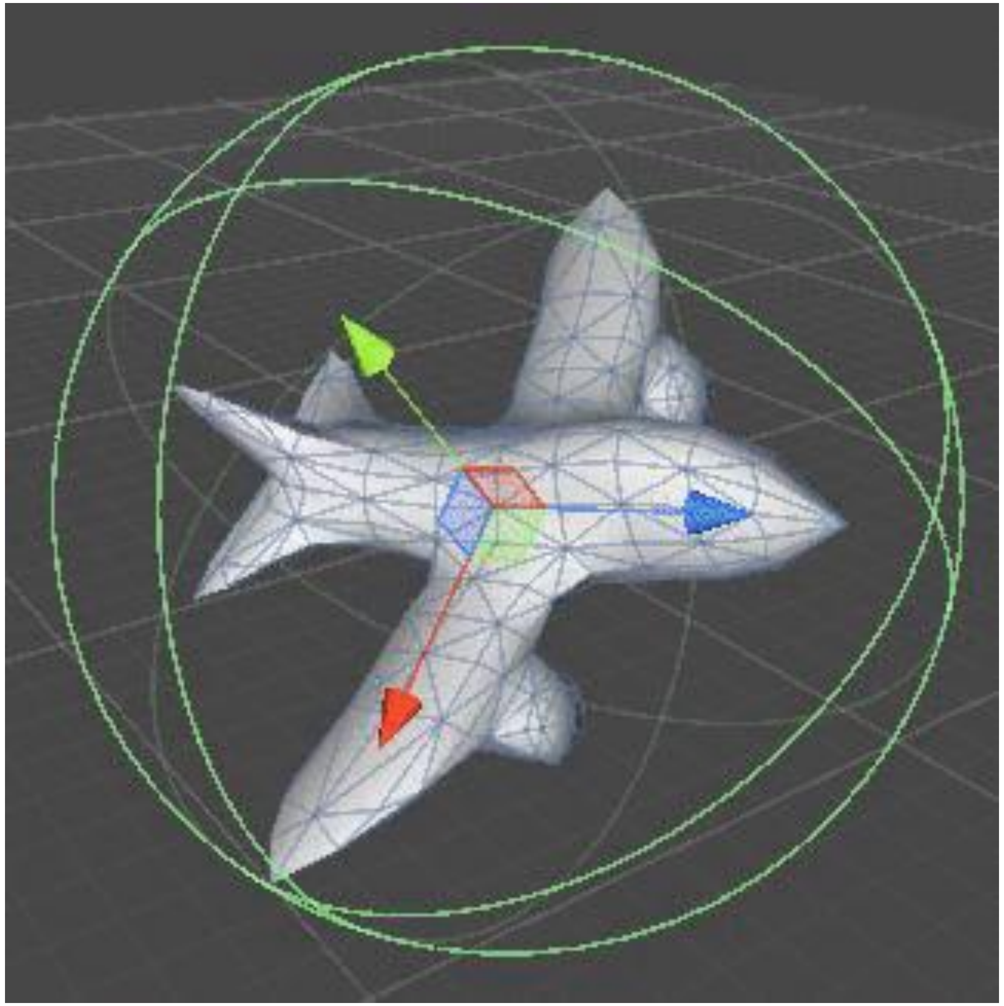
$\hat{\mathbf{u}}$ adalah vektor satuan dari vektor **u** = xi + y**j** + z**k**

$$\hat{\mathbf{u}} = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$$

dengan $||\hat{\mathbf{u}}|| = 1$



Sumber gambar: http://www.chrobotics.com/library/understanding-quaternions



Contoh 2: Misalkan sebuah titik P(0, 1, 1), atau sebagai vektor **p** = (0, 1, 1), diputar berlawanan arah jarum jam sejauh θ = 90° dengan sumbu rotasinya adalah **u** = *j*. Tentukan vektor bayangannya.

<u>Jawaban</u>:

u = **j**, panjangnya sama dengan satu, maka vektor satuannya juga sama yaitu $\hat{\mathbf{u}}$ = *j*

p = (0, 1, 1) = 0**i** + **j** + **k**

Nyatakan **p** dalam quaternion $\rightarrow$ *p* = 0 + 0i + j + k

$$q = \cos(\theta/2) + \sin(\theta/2)\,\hat{\mathbf{u}} = \cos 45^\circ + \sin 45^\circ(0i + j + 0k)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(0i + j + 0k) = \frac{\sqrt{2}}{2}(1 + 0i + j + 0k)$$

$$q^{-1} = \cos(\theta/2) - \sin(\theta/2)\,\hat{\mathbf{u}} = \cos 45^\circ - \sin 45^\circ(0i + j + 0k)$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(0i + j + 0k) = \frac{\sqrt{2}}{2}(1 - 0i - j - 0k)$$

Bayangan vektor **p** adalah **p'**:

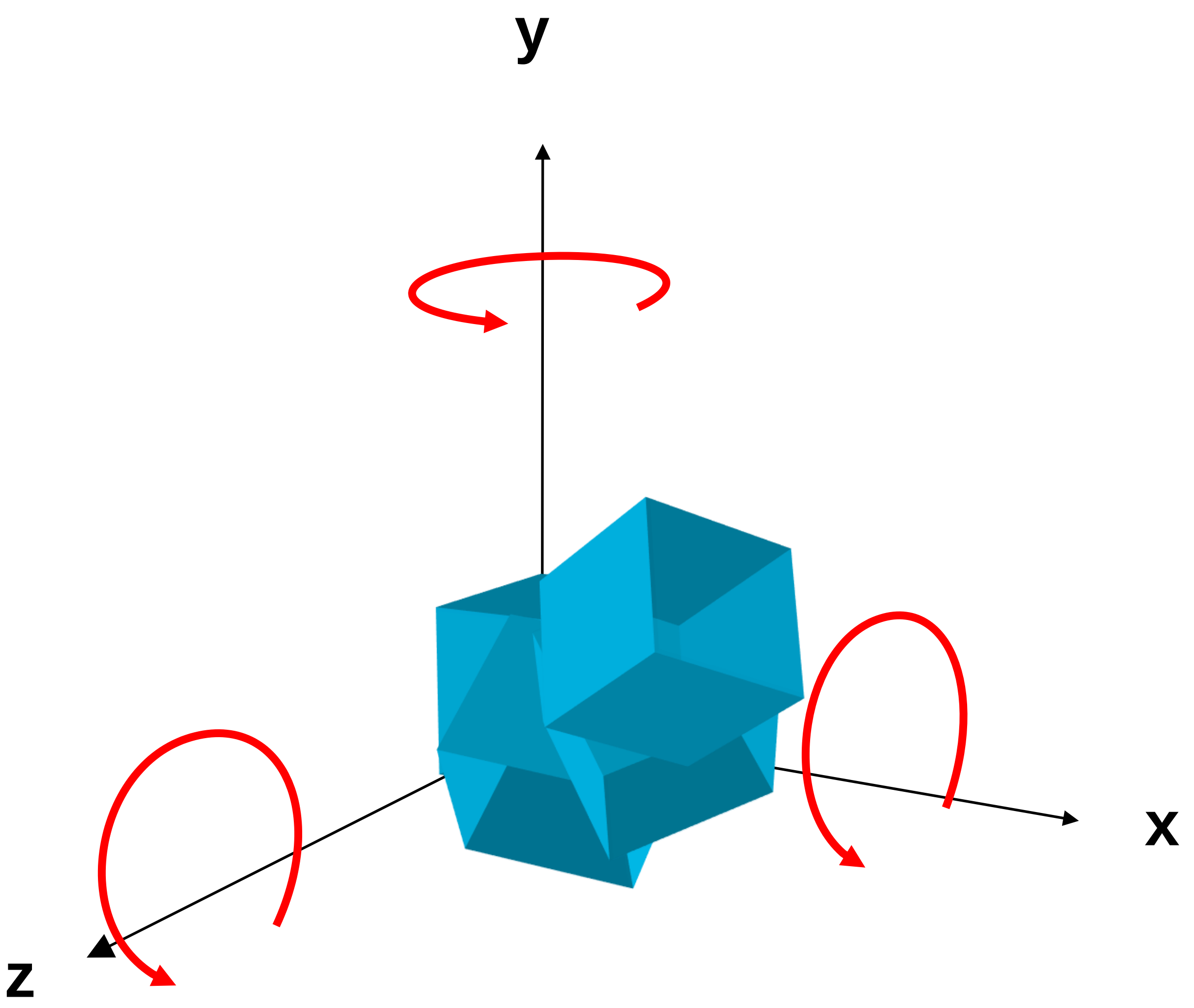
$$\mathbf{p}' = q\mathbf{p}q^{-1}$$

Dalam bentuk perkalian quaternion:

$$
\begin{aligned}
p' &= qpq^{-1} \\
&= \frac{\sqrt{2}}{2}(1 + 0i + j + 0k)(0 + 0i + j + k)\,\frac{\sqrt{2}}{2}(1 - 0i - j - 0k) \\
&= \frac{\sqrt{2}}{2}(-1 \quad + i + j + k)\,\frac{\sqrt{2}}{2}(1 - 0i - j - 0k) \\
&= \frac{1}{2}(-1 \; + 1 + j + i + j + k \; + i - k) \\
&= \frac{1}{2}(0 \; + 2i + 2j + 0k) \\
&= 0 + i + j + 0k
\end{aligned}
$$

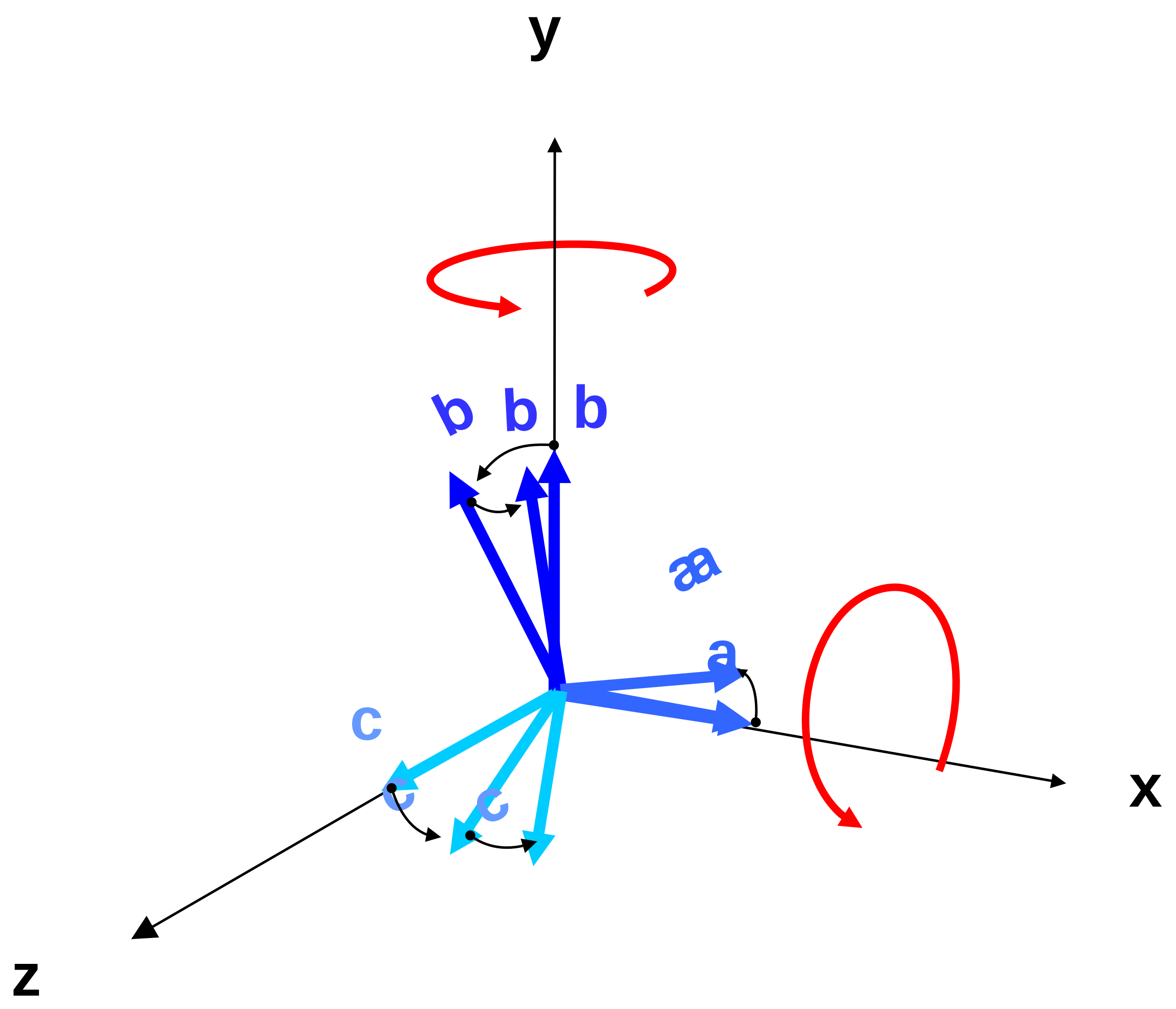
Jadi, **p'** = (1, 1, 0) = **i** + **j**

Let's do rotation!

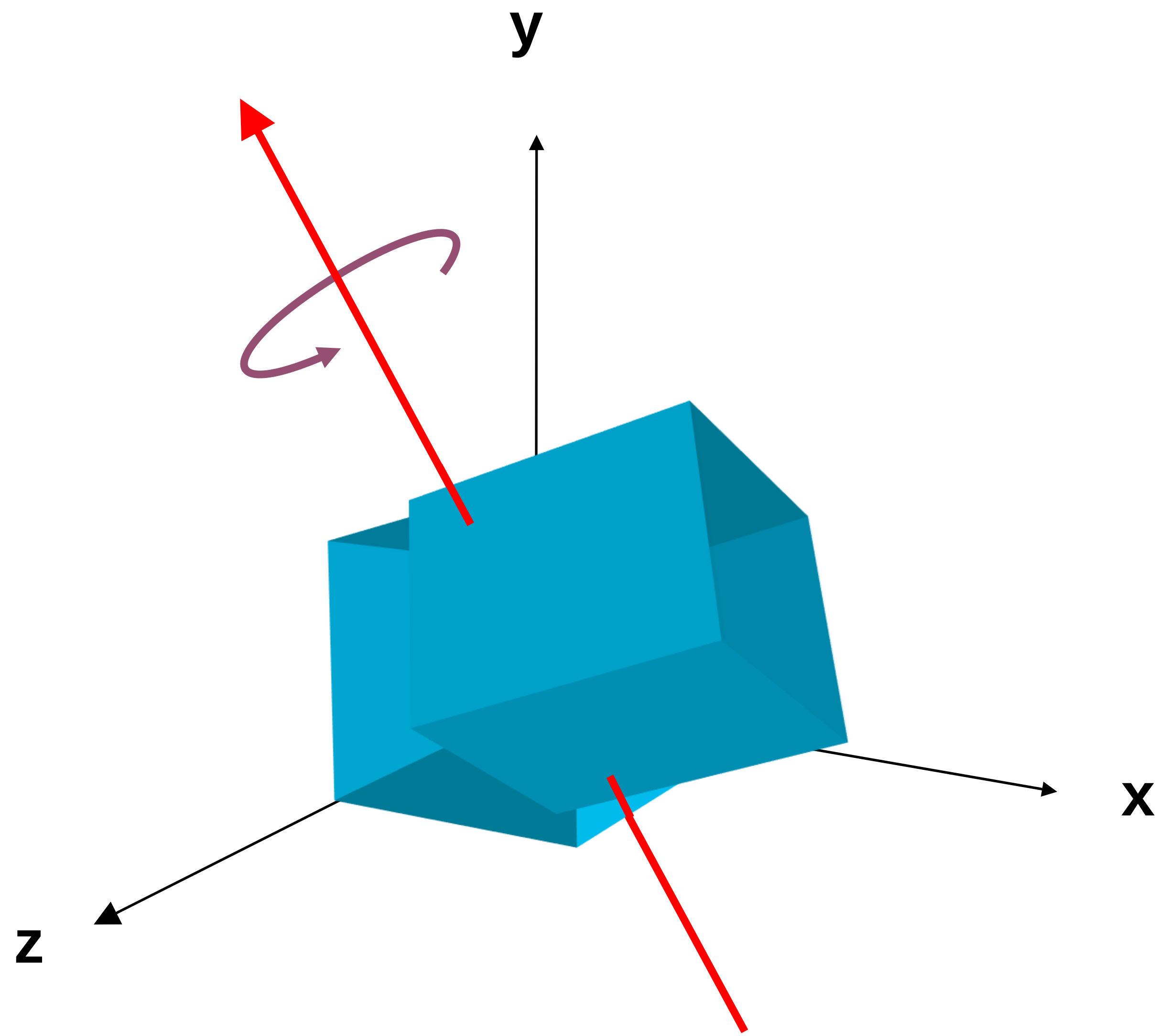


Sumber: Jyun-Ming Chen, 3D-Kinematics

Let's do rotation!



Let’s do another one!



Contoh 3 (Soal UAS 2019): Misalkan sebuah vektor **p** = 2**i** – 4**j** + 5**k** diputar berlawanan arah jarum jam sejauh θ = 120° dengan sumbu rotasinya adalah **u** = **i** + **j** + **k**. Tentukan vektor bayangannya.

<u>Jawaban</u>:

u = **i** + **j** + **k**, panjangnya = $\sqrt{3}$, maka vektor satuannya $\hat{\mathbf{u}} = \frac{1}{\sqrt{3}}$ (**i** + **j** + **k**)

p = (2, –4, 5) = 2**i** – 4**j** + 5**k**

Nyatakan **p** dalam quaternion $\rightarrow$ $p = 0 + 2i - 4j + 5k$

$$q = \cos(\theta/2) + \sin(\theta/2)\,\hat{\mathbf{u}} = \cos 60° + \sin 60°\left(\frac{1}{\sqrt{3}}(i + j + k)\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}(i + j + k)\right) = \frac{1}{2} + \frac{1}{2}(i + j + k) = \frac{1}{2}(1 + i + j + k)$$

$$q^{-1} = \cos(\theta/2) - \sin(\theta/2)\,\hat{\mathbf{u}} = \cos 60° - \sin 60°\left(\frac{1}{\sqrt{3}}(i + j + k)\right)$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}(i + j + k)\right) = \frac{1}{2}(1 - i - j - k)$$

Bayangan vektor **p** adalah **p'**:

$$\mathbf{p}' = q\mathbf{p}q^{-1}$$

Dalam bentuk perkalian quaternion:

$$p' = qpq^{-1}$$

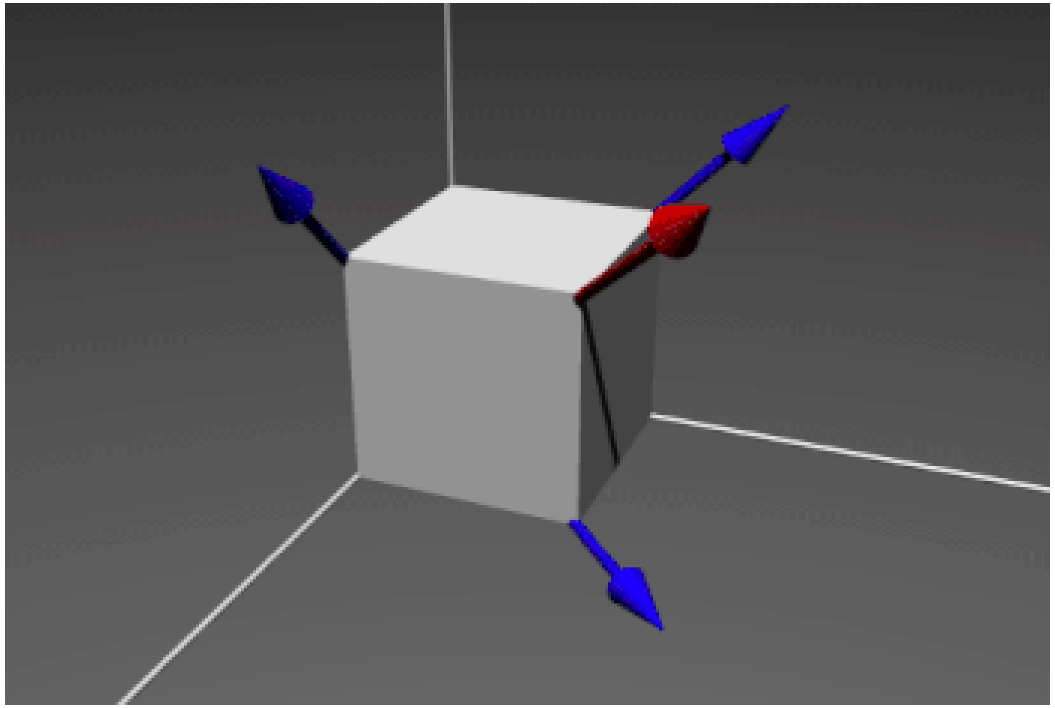
$$= \frac{1}{2}(1 + i + j + k)(0 + 2i - 4j + 5k)\,\frac{1}{2}\,(1 - i - j - k)$$

$$= \frac{1}{2}\,(11i - 7j - k - 3)\,\frac{1}{2}\,(1 - i - j - k)$$

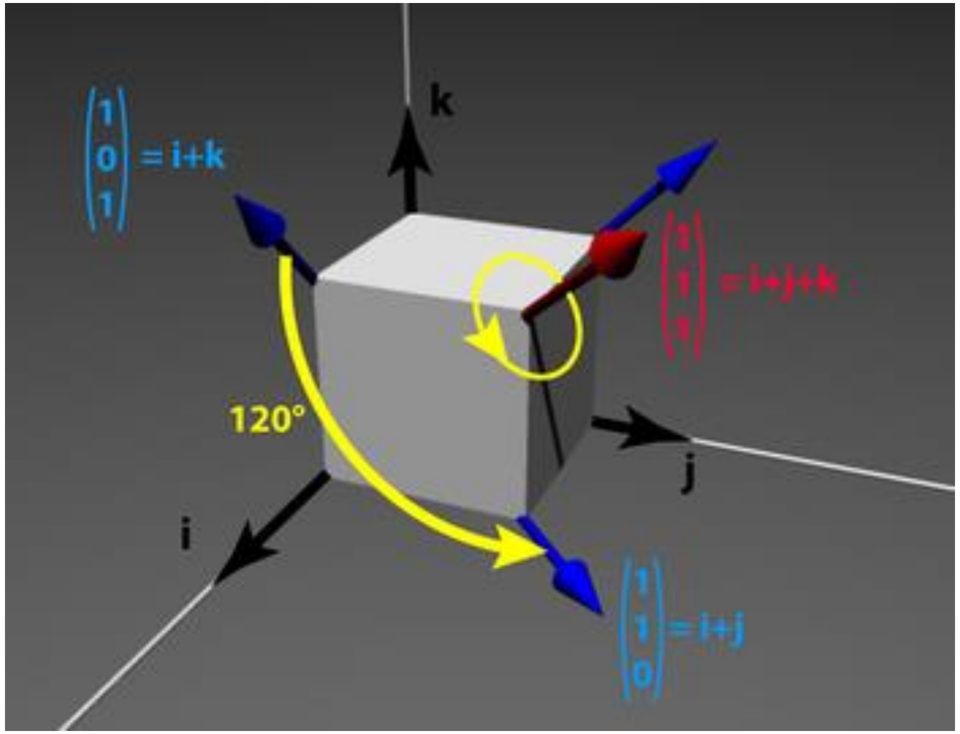
$$= \frac{1}{4}(20i + 8j - 16k + 0)$$

$$= 0 + 5i + j - 4k$$

Jadi, **p'** = (5, 1, –4) = 5**i** + **j** – 4**k**



A rotation around its diagonal



Position and Landmark

v = i + k

Sumbu putar: **u = i +j + k**
Sudut putaran = 120°

v' = i + j

Sumber: https://gepettoweb.laas.fr/hpp/pinocchio/doxygen-html/md_doc_c-maths_b-rigid-bodies.html

Latihan 2

(Soal UAS 2015)

Diketahui sebuah titik P=(1,1,1) diputar terhadap sumbu $\mathbf{u} = j + k$ sebesar 180^o, tentukan koordinat titik P' yang merupakan hasil dari rotasi tersebut.

Quaternion di dalam Bahasa Python

- **Instalasi paket** `pyquaternion`**:**

```
pip install pyquaternion
```

- **Gunakan paket** `pyquaternion`**:**

```
from pyquaternion import Quaternion
```

- **Buat (*create*) objek quaternion, ada banyak cara:**

```
q1 = Quaternion(scalar=1.0, vector=(0., 0., 0.))
q2 = Quaternion(scalar=1.0, vector=[0., 0., 0.])
q3 = Quaternion(scalar=1.0, vector=np.array([0., 0., 0.]))
q4 = Quaternion([1., 0., 0., 0.])
q5 = Quaternion((1., 0., 0., 0.))
q6 = Quaternion(np.array([1.0, 0., 0., 0.]))
```

Semuanya menghasilkan quaternion: q = 1 + 0i + 0j + 0k

- **Hitung norma (magnitude) quaternion**

```
q7 = Quaternion(np.array([1., 0., 0., 0.]))
print(q7.norm)
1.0
print(q7.magnitude)
1.0
```

- **Hitung balikan (inverse) quaternion**

```
print(q7.inverse)
1.000 -0.000i -0.000j -0.000k
```

- **Hitung *conjugate* quaternion**

```
print(q7.inverse)
1.000 -0.000i -0.000j -0.000k
```

- **Hitung quaternion satuan (atau menormalisasi quaternion):**

```
q8 = Quaternion(np.array([2.0, 1., 1., 1.]))
print(q8.normalised)
0.756 +0.378i +0.378j +0.378k
```

Contoh kode program rotasi vector dengan quaternion

```
import roslib
roslib.load_manifest('tf')
import tf from tf.transformations import *

# we want to rotate the point using the x-axis ("roll")
rot=quaternion_from_euler(-numpy.pi/4,0,0)

# the object is located in the y=1. We use the format [x,y,z,w] and w is allways 0 for
# vectors
vec= [0,1,0,0]

#now we apply the mathematical operation res=q*v*q'. We use this #function
#for multiplication but internally it is just a complex #multiplication #operation.
result=quaternion_multiply(quaternion_multiply(rot,  vec),quaternion_conjugate(rot))
```

Sumber kode: https://geus.wordpress.com/2012/02/16/basic-rotations-with-quaternions/