Seri bahan kuliah Algeo #25

Perkalian Geometri
(Bagian 3)

Bahan kuliah IF2123 Aljabar Linier dan Geometri

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Sumber:

John Vince, *Geometric Algebra for Computer Graphics*. Springer. 2007

Perkalian vektor dan bivector di R^3

- Diberikan vektor di R^3: $a = a_1 e_1 + a_2 e_2 + a_3 e_3$
  dan bivector: $B = b \wedge c$

- Perkalian geometri *a* dan *B* adalah (pembuktiannya tidak ditunjukkan di sini):

$$aB = a \cdot B + a \wedge B$$

- Perkalian geometri *B* dan *a* adalah (pembuktiannya tidak ditunjukkan di sini):

$$Ba = B \cdot a + B \wedge a$$

- Hubungan keduanya adalah:

$$a \cdot B = \frac{1}{2}(aB - Ba)$$

$$a \wedge B = \frac{1}{2}(aB + Ba)$$

Contoh 1: Diberikan tiga buah vektor di R^3 sebagai berikut

$$a = 2e_1 + e_2 - e_3$$

$$b = e_1 - e_2 + e_3$$

$$c = 2e_1 + 2e_2 + e_3.$$

Hitunglah (i) $B = b \wedge c$ (ii) aB (iii) Ba (iv) $a \cdot B$ (v) $a \wedge B$

<u>Jawaban</u>:

(i) $$B = b \wedge c = (e_1 - e_2 + e_3) \wedge (2e_1 + 2e_2 + e_3)$$

$$= 2e_{12} - e_{31} + 2e_{12} - e_{23} + 2e_{31} - 2e_{23}$$

$$B = 4e_{12} - 3e_{23} + e_{31}.$$

(ii) $$aB = (2e_1 + e_2 - 2e_3)(4e_{12} - 3e_{23} + e_{31})$$
$$= 8e_2 - 6e_{123} - 2e_3 - 4e_1 - 3e_3 + e_{123} - 8e_{123} - 6e_2 - 2e_1$$
$$aB = -6e_1 + 2e_2 - 5e_3 - 13e_{123}.$$ $\rightarrow$ vektor + trivector

(iii) $$Ba = (4e_{12} - 3e_{23} + e_{31})(2e_1 + e_2 - 2e_3)$$
$$= -8e_2 + 4e_1 - 8e_{123} - 6e_{123} + 3e_3 + 6e_2 + 2e_3 + e_{123} + 2e_1$$
$$Ba = 6e_1 - 2e_2 + 5e_3 - 13e_{123}.$$ $\rightarrow$ vektor + trivector

(iv) $$a \cdot B = \frac{1}{2}(aB - Ba)$$
$$= \frac{1}{2}(-6e_1 + 2e_2 - 5e_3 - 13e_{123} - 6e_1 + 2e_2 - 5e_3 + 13e_{123})$$
$$= \frac{1}{2}(-12e_1 + 4e_2 - 10e_3)$$
$$a \cdot B = -6e_1 + 2e_2 - 5e_3.$$ $\rightarrow$ vektor

(v) $$a \wedge B = \frac{1}{2}(aB + Ba)$$

$$= \frac{1}{2}(-6e_1 + 2e_2 - 5e_3 - 13e_{123} + 6e_1 - 2e_2 + 5e_3 - 13e_{123})$$

$$a \wedge B = -13e_{123}.$$ → trivector

Dari (iv) dan (v) terlihat bahwa:

$$aB = a \cdot B + a \wedge B$$

$$aB = -6e_1 + 2e_2 - 5e_3 - 13e_{123}.$$

yng berarti bahwa *aB* diidentifikasi oleh *inner product* ($a \cdot B$) dan *outer product* ($a \wedge B$)

Perkalian *bivector-bivector* satuan di R^3

$$e_{12}^2 = e_{23}^2 = e_{31}^2 = -1$$

$$e_{12}e_{23} = e_{13} = -e_{31}$$

$$e_{23}e_{31} = e_{21} = -e_{12}$$

$$e_{31}e_{12} = e_{32} = -e_{23}$$

$$e_{12}e_{31} = e_{23}$$

$$e_{23}e_{12} = e_{31}$$

$$e_{31}e_{23} = e_{12}.$$

TABLE 8.4

GP	e_{12}	e_{23}	e_{31}
e_{12}	-1	$-e_{31}$	e_{23}
e_{23}	e_{31}	-1	$-e_{12}$
e_{31}	$-e_{23}$	e_{12}	-1

Contoh cara mendapatkan salah satu hasil di samping:

$$\begin{aligned} e_{31}e_{23} &= e_3e_1e_2e_3 \\ &= -e_3e_1e_3e_2 \\ &= e_3e_3e_1e_2 \\ &= e_3^2e_1e_2 \\ &= (1)e_1e_2 = e_1e_2 = e_{12} \end{aligned}$$

Perkalian vektor dan *trivector* di R^3

Perkalian vektor dengan *trivector* menghasilkan *bivector*

$$e_1 e_{123} = e_{23} \qquad e_{123} e_1 = e_{23}$$

$$e_2 e_{123} = e_{31} \quad \Longleftrightarrow \quad e_{123} e_2 = e_{31}$$

$$e_3 e_{123} = e_{12}. \qquad e_{123} e_3 = e_{12}.$$

$\therefore$ Perkalian vektor dengan *trivector* bersifat komutatif

Contoh 2: Diberikan vektor $a = 2e_1 + 3e_2 + 4e_3$ dan trivector $B = 5(e_1 \wedge e_2 \wedge e_3) = 5e_{123}$

Hitunglah *aB*.

<u>Jawaban</u>:

$$
\begin{aligned}
aB &= (2e_1 + 3e_2 + 4e_3)\, 5e_{123} \\
&= 10e_1e_{123} + 15e_2e_{123} + 20e_3e_{123} \\
&= 10e_1e_1e_2e_3 + 15e_2e_1e_2e_3 + 20e_3e_1e_2e_3 \\
&= 10e_2e_3 - 15e_2e_2e_1e_3 - 20e_3e_1e_3e_2 \\
&= 10e_2e_3 - 15e_1e_3 + 20e_3e_3e_1e_2 \\
&= 10e_2e_3 + 15e_3e_1 + 20e_1e_2 \\
&= 20e_1e_2 + 10e_2e_3 + 15e_3e_1 \\
&= 20e_{12} + 10e_{23} + 15e_{31}
\end{aligned}
$$

Perkalian vektor dengan *trivector* menghasilkan tiga buah *bivector*.

Perkalian *bivector* dan *trivector* di R^3

Perkalian *bivector* dengan *trivector* menghasilkan vector

$$e_{12}e_{123} = -e_3 \qquad e_{123}e_{12} = -e_3$$

$$e_{23}e_{123} = -e_1 \qquad e_{123}e_{23} = -e_1$$

$$e_{31}e_{123} = -e_2. \qquad e_{123}e_{31} = -e_2.$$

$\therefore$ Perkalian bivector dengan *trivector* bersifat komutatif

Contoh 3 : Diberikan *bivector* $B = 2e_{12} + 3e_{23} + 4e_{31}$ dan *trivector* $C = 5e_{123}$
Hitunglah BC.

Jawaban:

$$B5e_{123} = (2e_{12} + 3e_{23} + 4e_{31})5e_{123}$$
$$= -15e_1 - 20e_2 - 10e_3.$$

Ringkasan perkalian vektor di R^3

TABLE 8.5

Inner product	
Vectors commute	$a \cdot b = b \cdot a$
Vectors and bivectors anticommute	$a \cdot B = -B \cdot a$
	$a \cdot B = \frac{1}{2}(aB - Ba)$
	$a \cdot B = (a \cdot b)c - (a \cdot c)b$
	$B \cdot a = \frac{1}{2}(Ba - aB)$
	$B \cdot a = (a \cdot c)b - (a \cdot b)c$
Outer product	
Vectors anticommute	$a \wedge b = -b \wedge a$
Vectors and bivectors commute	$a \wedge B = B \wedge a$
	$a \wedge B = \frac{1}{2}(aB + Ba)$
	$a \wedge B = abc$
	$B \wedge a = \frac{1}{2}(Ba + aB)$
	$B \wedge a = abc$

Geometric product	
Orthogonal vectors anticommute	$e_{12} = -e_{21}$
Orthogonal bivectors anticommute	$e_{12}e_{23} = -e_{23}e_{12}$
Bivectors square to -1	$e_{12}^2 = e_{23}^2 = e_{31}^2 = -1$
Definition	$ab = a \cdot b + a \wedge b$
Vectors and bivectors anticommute	$aB = -Ba$
	$aB = a \cdot B + a \wedge B$
	$aB = (a \cdot b)c - (a \cdot c)b + abc$
	$Ba = B \cdot a + B \wedge a$
	$Ba = (a \cdot c)b - (a \cdot b)c + abc$
Trivector commutes with all multivectors in the space	$aT = Ta \quad BT = TB$
The pseudoscalar	$e_{123} = I$
Vectors and the pseudoscalar commute	$aI = Ia$
	$aI = a \cdot I$
Duality transformation	$e_{23} = Ie_1$
	$e_{31} = Ie_2$
	$e_{12} = Ie_3$
The trivector squares to -1	$I^2 = -1$
Where a and b are vectors, B is a bivector, and T is a trivector.	

TABLE 8.6

GP	λ	e_1	e_2	e_3	e_{12}	e_{23}	e_{31}	e_{123}
λ	λ^2	λe_1	λe_2	λe_3	λe_{12}	λe_{23}	λe_{31}	λe_{123}
e_1	λe_1	1	e_{12}	$-e_{31}$	e_2	e_{123}	$-e_3$	e_{23}
e_2	λe_2	$-e_{12}$	1	e_{23}	$-e_1$	e_3	e_{123}	e_{31}
e_3	λe_3	e_{31}	$-e_{23}$	1	e_{123}	$-e_2$	e_1	e_{12}
e_{12}	λe_{12}	$-e_2$	e_1	e_{123}	-1	$-e_{31}$	e_{23}	$-e_3$
e_{23}	λe_{23}	e_{123}	$-e_3$	e_2	e_{31}	-1	$-e_{12}$	$-e_1$
e_{31}	λe_{31}	e_3	e_{123}	$-e_1$	$-e_{23}$	e_{12}	-1	$-e_2$
e_{123}	λe_{123}	e_{23}	e_{31}	e_{12}	$-e_3$	$-e_1$	$-e_2$	-1

Keterangan: GP = Geometry Product

Balikan (*inverse*) vektor

- Pada aljabar elementer, $c = ab$ (a dan b bilangan riil) maka $a = cb^{-1}$ atau $b^{-1} = \frac{a}{c}$ (syarat $c \neq 0$)

- Pada aljabar geometri, $B = ab$ (a dan b vektor, B multivektor), maka

$$Bb = (ab)b \qquad \text{(kalikan kedua ruas dengan } b\text{)}$$
$$= ab^2$$
$$a = B\frac{b}{b^2} = B\,\frac{1}{b} = Bb^{-1}$$
$$a = Bb^{-1}$$

yang dalam hal ini

$$b^{-1} = \frac{b}{b^2} = \frac{b}{\|b\|^2}$$

→ balikan vektor *b*

Contoh 4: Diberikan vektor *a* dan *b*: $a = 3e_1 + 4e_2$ dan $b = e_1 + e_2$

Hitung *B* = *ab* dan balikan vektor *b*.

Jawaban:

(i) $B = ab = (3e_1 + 4e_2)(e_1 + e_2) = 3e_1^2 + 3e_1e_2 + 4e_2e_1 + 4e_2^2$

$= 3 + 3e_{12} - 4e_{12} + 4 = 7 - e_{12}$

(atau pakai rumus: $ab = a \cdot b + a \wedge b = (3)(1) + (4)(1) + \{(3)(1) - (4)(1)\}\, e_1 \wedge e_2$

$= 7 - e_{12}$)

(ii) $b^{-1} = \dfrac{b}{\|b\|^2} = \dfrac{e_1 + e_2}{(\sqrt{1^2+1^2})^2} = \dfrac{e_1 + e_2}{2} = \dfrac{1}{2}(e_1 + e_2)$

Periksa bahwa *a* dapat diperoleh kembali sebagai berikut:

$$a = Bb^{-1} = \frac{1}{2}(7 - e_{12})(e_1 + e_2) = \frac{1}{2}(7e_1 + 7e_2 - e_{12}e_1 - e_{12}e_2)$$

$$= \frac{1}{2}(7e_1 + 7e_2 + e_2 - e_1) = 3e_1 + 4e_2$$

Operasi *meet*

- Operasi *meet* bertujuan untuk mencari perpotongan garis, bidang, volume, dll. lain.
- Notasi: $A \vee B$
- Operasi *meet* didefinisikan sebagai berikut:

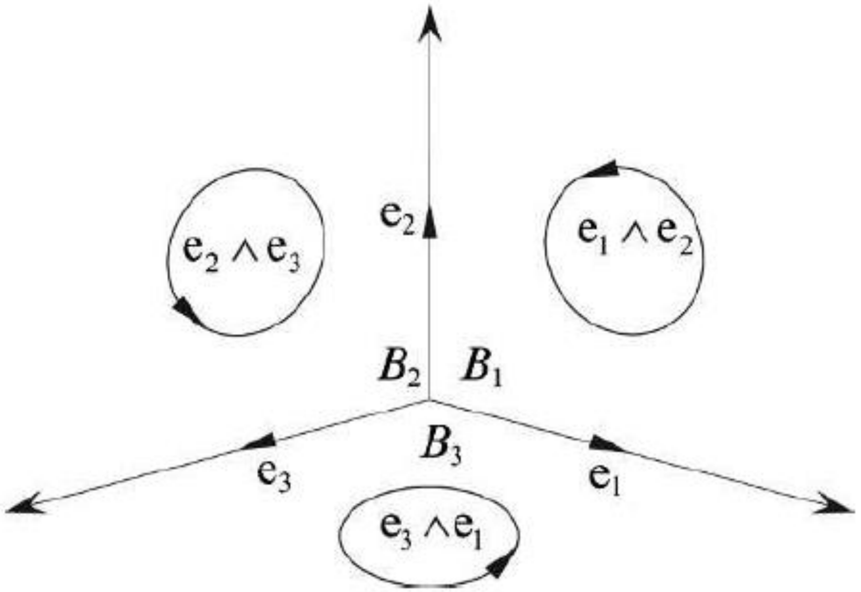
$$A \vee B = A^* \cdot B.$$

yang dalam hal ini, A^* = pseudoscalar dari $A = IA$

Contoh: (i) $A = e_3 \rightarrow A^* = IA = e_{123}e_3 = e_{12}$

(ii) $A = 2e_1 + e_3 \rightarrow A^* = IA = e_{123}(2e_1 + e_3) = 2e_{123}e_1 + e_{123}e_3 = 2e_{23} + e_{12}$

Perhatikan tiga bilah B_1, B_2, dan B_3 yang dibentuk oleh vektor-vektor satuan



$$B_1 = e_1 \wedge e_2$$

$$B_2 = e_2 \wedge e_3$$

$$B_3 = e_3 \wedge e_1$$

Perpotongan bilah B_1 dan B_2 adalah sumbu e_2 → $B_1 \vee B_2 = e_2$

Perpotongan bilah B_2 dan B_3 adalah sumbu e_3 → $B_2 \vee B_3 = e_3$

Perpotongan bilah B_1 dan B_3 adalah sumbu e_1 → $B_3 \vee B_1 = e_1$

- Akan ditunjukkan bahwa $B_1 \vee B_2 = e_2$ dengan operasi *meet*:

$$B_1 \vee B_2 = B_1^* \cdot B_2$$
$$= (e_{123}e_{12}) \cdot e_{23} = (e_1e_2e_3e_1e_2) \cdot e_{23} = (-e_1^2e_2^2e_3) \cdot e_{23}$$
$$= -e_3 \cdot e_{23}$$

dengan mengingat bahwa $a \cdot B = \frac{1}{2}(aB - Ba)$ maka $\underset{a}{-e_3} \cdot \underset{B}{e_{23}}$ dapat dinyatakan sebagai

$$-e_3 \cdot e_{23} = \frac{1}{2}(-e_{323} + e_{233})$$

sehingga

$$B_1 \vee B_2 = \frac{1}{2}(-e_{323} + e_{233}) = \frac{1}{2}(e_3e_3e_2 + e_2e_3e_3)$$
$$= \frac{1}{2}(e_3^2e_2 + e_2e_3^2) = \frac{1}{2}(e_2 + e_2) = e_2$$

(terbukti)

- Dengan cara yang sama, maka

$$
\begin{aligned}
B_2 \vee B_3 &= B_2^* \cdot B_3 \\
&= (e_{123}e_{23}) \cdot e_{31} \\
&= -e_1 \cdot e_{31} \\
&= \frac{1}{2}(-e_{131} + e_{311}) \\
B_2 \vee B_3 &= e_3
\end{aligned}
$$

dan

$$
\begin{aligned}
B_3 \vee B_1 &= B_3^* \cdot B_1 \\
&= (e_{123}e_{31}) \cdot e_{12} \\
&= -e_2 \cdot e_{12} \\
&= \frac{1}{2}(-e_{212} + e_{122}) \\
B_3 \vee B_1 &= e_1.
\end{aligned}
$$

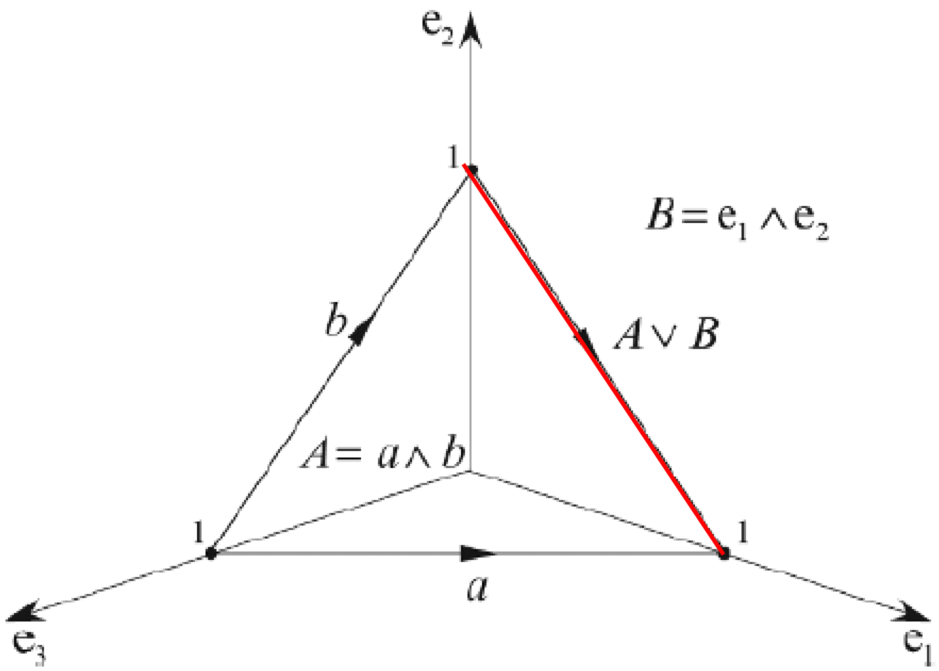
Contoh 5: Diberikan dua buah vektor *a* dan *b* sebagai berikut:

$$a = e_1 - e_3$$

$$b = e_2 - e_3$$

Tentukan perpotongan bidang *A* dan *B*, yang dalam hal ini $A = a \wedge b$ dan $B = e_{12}$

Jawaban:



$$A = a \wedge b$$

$$= (e_1 - e_3) \wedge (e_2 - e_3) = e_{12} - e_{13} - e_{32}$$

$$A \vee B = A^* \cdot B$$

$$= e_{123}(e_{12} - e_{13} - e_{32}) \cdot e_{12} = (-e_3 - e_2 - e_1) \cdot e_{12}$$

Gunakan $a \cdot B = \frac{1}{2}(aB - Ba)$ maka

$$A \vee B = \frac{1}{2}((-e_3 - e_2 - e_1)e_{12} - e_{12}(-e_3 - e_2 - e_1))$$

$$= e_1 - e_2$$ (garis yang berwarna merah)

Latihan (Soal UAS 2019)

Diberikan tiga buah vektor sebagai berikut:

$$a = 2e_1 + e_2 - e_3$$

$$b = e_1 - e_2 - e_3$$

$$c = 2e_1 + 2e_2 - e_3$$

a) Jika B adalah multivektor, $B = ab$, perlihatkan bahwa $a = Bb^{-1}$

b) Tentukan perpotongan bidang yang dibentuk oleh vektor *b* dan *c* dengan bidang $(e_2 \wedge e_3)$

Jawaban:

(a) $B = ab = (2e_1 + e_2 - e_3)(e_1 - e_2 - e_3) = 2 - 3e_{12} - 2e_{23} + e_{31}$

$$b^{-1} = \frac{b}{\|b\|^2} = \frac{e_1 - e_2 - e_3}{(\sqrt{1^2+(-1)^2+(-1)^2})^2} = \frac{e_1 - e_2 - e_3}{3} = \frac{1}{3}(e_1 - e_2 - e_3)$$

sehingga

$$Bb^{-1} = (2 - 3e_{12} - 2e_{23} + e_{31})\,\frac{1}{3}(e_1 - e_2 - e_3)$$

$$= \frac{1}{3}((2e_1 - 2e_2 - 2e_3) + (3e_2 + 3e_1 + 3e_{123}) +$$

$$(-2e_{123} - 2e_3 + 2e_2) + (e_3 - 2e_{123} + e_1))$$

$$= \frac{1}{3}(6e_1 + 3e_2 - 3e_3) = 2e_1 + e_2 - e_3 = a$$ (terbukti)

(b) Bidang yang dibentuk oleh *b* dan *c* misalkan adalah *A*

$A = b \wedge c = (e_1 - e_2 - e_3) \wedge (2e_1 + 2e_2 - e_3)$

$= 4e_{12} + 3e_{23} - e_{31}$

Maka, perpotongan *A* dengan $e_2 \wedge e_3$ dihitung dengan operasi *meet*:

$A \vee (e_2 \wedge e_3) = A^* \cdot (e_2 \wedge e_3)$

$= e_{123}(4e_{12} + 3e_{23} - e_{31}) \cdot e_{23}$ (Ket: $A^* = IA = e_{123}A$)

$= 4e_{12312} + 3e_{12323} - e_{12331}) \cdot e_{23}$

$= (-4e_3 - 3e_1 + e_2) \cdot e_{23}$ Gunakan $a \cdot B = \frac{1}{2}(aB - Ba)$

$= \frac{1}{2}((-4e_3 - 3e_1 + e_2)e_{23} - e_{23}(-4e_3 - 3e_1 + e_2))$

$= \frac{1}{2}(-4e_{323} - 3e_{123} + e_{223} + 4e_{233} + 3e_{231} - e_{232})$

$= \frac{1}{2}(8e_2 + 2e_3)$

$= 4e_2 + e_3$

Multivector di R^3

- *Multivector* di R^3 mengandung skalar, vektor, *bivector*, dan trivector.
- *Multivector* di R^3 merupakan kombinasi linier dari skalar, vektor, *bivector*, dan trivector. Elemen-elemen di dalam *multivector* diresumekan pada tabel berikut:

TABLE 8.7

Element	Symbol	Grade
1 scalar	λ	0
3 vectors	$\{e_1, e_2, e_3\}$	1
3 bivectors	$e_1 \wedge e_2 = e_{12}$	2
	$e_2 \wedge e_3 = e_{23}$	
	$e_3 \wedge e_1 = e_{31}$	
1 trivector	e_{123}	3

- Diberikan dua buah multivector di R^3 sebagai berikut:

$$A = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_{12} + \lambda_4 e_{23} + \lambda_5 e_{31} + \lambda_6 e_{123} \quad [\lambda_i \in \mathbb{R}]$$

$$B = \beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123} \quad [\beta_i \in \mathbb{R}]$$

- Perkalian geometri *A* dan *B* adalah sebagai berikut:

$$\begin{aligned} AB = \lambda_0(&\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ &+ \lambda_1 e_1(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ &+ \lambda_2 e_2(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ &+ \lambda_3 e_{12}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ &+ \lambda_4 e_{23}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ &+ \lambda_5 e_{31}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ &+ \lambda_6 e_{123}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \end{aligned}$$

$$
\begin{aligned}
AB = {} & \lambda_0\beta_0 + \lambda_0\beta_1 e_1 + \lambda_0\beta_2 e_2 + \lambda_0\beta_3 e_{12} + \lambda_0\beta_4 e_{23} + \lambda_0\beta_5 e_{31} + \lambda_0\beta_6 e_{123} + \lambda_1\beta_0 e_1 + \lambda_1\beta_1 \\
& + \lambda_1\beta_2 e_{12} + \lambda_1\beta_3 e_2 + \lambda_1\beta_4 e_{123} - \lambda_1\beta_5 e_3 + \lambda_1\beta_6 e_{23} + \lambda_2\beta_0 e_2 - \lambda_2\beta_1 e_{12} + \lambda_2\beta_2 \\
& - \lambda_2\beta_3 e_1 + \lambda_2\beta_4 e_3 + \lambda_2\beta_5 e_{123} + \lambda_2\beta_6 e_{31} + \lambda_3\beta_0 e_{12} - \lambda_3\beta_1 e_2 + \lambda_3\beta_2 e_1 - \lambda_3\beta_3 \\
& - \lambda_3\beta_4 e_{31} + \lambda_3\beta_5 e_{23} - \lambda_3\beta_6 e_3 + \lambda_4\beta_0 e_{23} + \lambda_4\beta_1 e_{123} - \lambda_4\beta_2 e_3 + \lambda_4\beta_3 e_{31} - \lambda_4\beta_4 \\
& - \lambda_4\beta_5 e_{12} - \lambda_4\beta_6 e_1 + \lambda_5\beta_0 e_{31} + \lambda_5\beta_1 e_3 + \lambda_5\beta_2 e_{123} - \lambda_5\beta_3 e_{23} + \lambda_5\beta_4 e_{12} - \lambda_5\beta_5 \\
& - \lambda_5\beta_6 e_2 + \lambda_6\beta_0 e_{123} + \lambda_6\beta_1 e_{23} + \lambda_6\beta_2 e_{31} - \lambda_6\beta_3 e_3 - \lambda_6\beta_4 e_1 - \lambda_6\beta_5 e_2 - \lambda_6\beta_6 \qquad (8.238) \\
= {} & \lambda_0\beta_0 + \lambda_1\beta_1 + \lambda_2\beta_2 - \lambda_3\beta_3 - \lambda_4\beta_4 - \lambda_5\beta_5 - \lambda_6\beta_6 \\
& + (\lambda_0\beta_1 + \lambda_1\beta_0 - \lambda_2\beta_3 + \lambda_3\beta_2 - \lambda_4\beta_6 - \lambda_6\beta_4)e_1 \\
& + (\lambda_0\beta_2 + \lambda_1\beta_3 + \lambda_2\beta_0 - \lambda_3\beta_1 - \lambda_5\beta_6 - \lambda_6\beta_5)e_2 \\
& + (-\lambda_1\beta_5 + \lambda_2\beta_4 - \lambda_3\beta_6 - \lambda_4\beta_2 + \lambda_5\beta_1 - \lambda_6\beta_3)e_3 \\
& + (\lambda_0\beta_3 + \lambda_1\beta_2 - \lambda_2\beta_1 + \lambda_3\beta_0 - \lambda_4\beta_5 + \lambda_5\beta_4)e_{12} \\
& + (\lambda_0\beta_4 + \lambda_1\beta_6 + \lambda_3\beta_5 + \lambda_4\beta_0 - \lambda_5\beta_3 + \lambda_6\beta_1)e_{23} \\
& + (\lambda_0\beta_5 + \lambda_2\beta_6 - \lambda_3\beta_4 + \lambda_4\beta_3 + \lambda_5\beta_0 + \lambda_6\beta_2)e_{31} \\
& + (\lambda_0\beta_6 + \lambda_1\beta_4 + \lambda_2\beta_5 + \lambda_4\beta_1 + \lambda_5\beta_2 + \lambda_6\beta_0)e_{123}
\end{aligned}
$$

Hubungan antara aljabar vektor dengan aljabar geometri

TABLE 8.9

Vector Algebra		Geometric Algebra	
vector	$v = a_1\mathrm{i} + a_2\mathrm{j}$	vector	$v = a_1\mathrm{e}_1 + a_2\mathrm{e}_2$
map	$a = a_1 \quad b = a_2$	map	$Z = \mathrm{e}_1 v$
complex number	$z = a + bi$	multivector	$Z = a_1 + a_2 I$
rotor	$z' = ze^{i\phi}$	rotor	$Z' = Ze^{I\phi}$
			$v' = ve^{I\phi}$
90° rotor	$v' = -a_2\mathrm{i} + a_1\mathrm{j}$	90° rotor	$v' = vI$

Hubungan antara aljabar geometri dengan aljabar quaternion

Perkalian *i*, *j*, dan *k* di dalam aljabar quaternion:

TABLE 8.10

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

$$ijk = -1$$

Pada kedua tabel, hasil perkalian merupakan pencerminan terhadap diagonal utama namun dengan tanda berbeda

Misalkan didefinisikan tiga buah *bivector*:

$$B_1 = e_2 \wedge e_3$$

$$B_2 = e_3 \wedge e_1$$

$$B_3 = e_1 \wedge e_2$$

Perkalian ketiga buah *bivector* hasilnya:

TABLE 8.11

	B_1	B_2	B_3
B_1	-1	$-B_3$	B_2
B_2	B_3	-1	$-B_1$
B_3	$-B_2$	B_1	-1

$$B_1 B_2 B_3 = +1$$

Hubungan antara *outer product* dan *cross product*

- Diberikan dua buah vektor di R^3: $a = a_1 e_1 + a_2 e_2 + a_3 e_3$ dan $b = b_1 e_1 + b_2 e_2 + b_3 e_3$

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = (a_2 b_3 - a_3 b_2) e_1 + (a_3 b_1 - a_1 b_3) e_2 + (a_1 b_2 - a_2 b_1) e_3$$

$$a \wedge b = (a_2 b_3 - a_3 b_2) e_{23} + (a_3 b_1 - a_1 b_3) e_{31} + (a_1 b_2 - a_2 b_1) e_{12}.$$

- Kalikan e_{123} dengan $a \wedge b$:

$$e_{123}(a \wedge b) = (a_2b_3 - a_3b_2)e_{123}e_{23} + (a_3b_1 - a_1b_3)e_{123}e_{31} + (a_1b_2 - a_2b_1)e_{123}e_{12}$$

$$e_{123}(a \wedge b) = -(a_2b_3 - a_3b_2)e_1 - (a_3b_1 - a_1b_3)e_2 - (a_1b_2 - a_2b_1)e_3.$$

- Kalikan kedua ruas dengan –1:

$$-e_{123}(a \wedge b) = (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3$$

- Maka dapat dinyatakan bahwa:

$$a \times b = -e_{123}(a \wedge b) = -I(a \wedge b)$$

- Dengan mengingat bahwa $a \times b$ menghasilkan vektor v yang ortogonal dengan a dan b, maka

$$v = -IB \qquad (v = a \times b \text{ dan } B = a \wedge b)$$

Contoh 6: Diberikan dua buah vektor di R^3 sebagai berikut:

$$a = -e_2 + e_3$$

$$b = e_1 - e_2.$$

Maka perkalian silang *a* dan *b* adalah

$$a \times b = c = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= e_1 + e_2 + e_3$$

Vektor *c* dapat dihitung pula sebagai berikut

$$\begin{aligned} B &= a \wedge b \\ &= (-e_2 + e_3) \wedge (e_1 - e_2) \\ &= -e_2 \wedge e_1 + e_2 \wedge e_2 + e_3 \wedge e_1 - e_3 \wedge e_2 \\ B &= e_{12} + e_{31} + e_{23}. \end{aligned}$$

$$\begin{aligned} c &= -IB \\ &= -e_{123}(e_{12} + e_{31} + e_{23}) \\ &= e_3 + e_2 + e_1 \\ c &= e_1 + e_2 + e_3 \end{aligned}$$

(hasilnya sama)

TAMAT