THE GEOMETRIC PRODUCT (CH. 8)

Aljabar Geometri (IF2123)

Sekolah Teknik Elektro dan Informatika
Institut Teknologi Bandung

Clifford's definition of the geometric product

$$(a \cdot b)c \quad a \cdot (b \times c) \quad (a \times b) \times c \quad a \times (b \times c) \tag{8.1}$$

which are easy to interpret and visualize. On the other hand, GA employs a new product called the *geometric product*, which operates upon multivectors containing scalars, vectors, areas and volumes. Visualizing these products can be difficult. For example, how should we visualize the

Clifford defined the geometric product of two vectors a and b as

$$ab = a \cdot b + a \wedge b \tag{8.2}$$

which is the sum of a scalar and a bivector. Now there is always a good reason why such definitions are made, and it is far from arbitrary. In order to develop this new product we start by defining the axioms associated with the algebra. These comprise an associative axiom, distributive axiom, and a definition of a modulus.

Associative axiom

$$a(bc) = (ab)c = abc \tag{8.3}$$

$$(\lambda a)b = \lambda(ab) = \lambda ab \quad [\lambda \in \mathbb{R}]. \tag{8.4}$$

Distributive axiom

$$a(b + c) = ab + ac \tag{8.5}$$

and

$$(b + c)a = ba + ca. \tag{8.6}$$

Modulus

$$a^2 = \pm\|a\|^2. \tag{8.7}$$

From these axioms we can derive the meaning of the product ab. Just in case the product is antisymmetric, we pay particular attention to the order of vectors.

We begin with two vectors a and b and represent their sum as

$$c = a + b. \tag{8.8}$$

Therefore,

$$c^2 = (a + b)^2 \tag{8.9}$$

and

$$c^2 = a^2 + b^2 + ab + ba. \tag{8.10}$$

To simplify this relationship we investigate how Eq. (8.10) behaves when vectors a and b are orthogonal, linearly dependent and linearly independent.

Orthogonal vectors

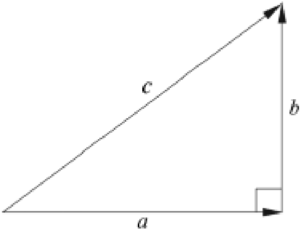

FIGURE 8.1.

With reference to Fig. 8.1, when

$$b \perp a$$

then

$$\|c\|^2 = \|a\|^2 + \|b\|^2. \tag{8.11}$$

Invoking the modulus axiom, we have

$$c^2 = a^2 + b^2 \tag{8.12}$$

which implies that in Eq. (8.10)

$$ab + ba = 0 \tag{8.13}$$

Linearly dependent vectors

With reference to Fig. 8.2, when

$$b \parallel a \quad \text{and} \quad b = \lambda a \quad \text{where} \quad [\lambda \in \mathbb{R}] \tag{8.15}$$

$$ab = a\lambda a = \lambda aa = ba \tag{8.16}$$

which confirms that linearly dependent vectors commute.

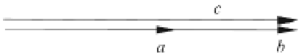

Figure 8.2.

Invoking the modulus axiom we have

$$\lambda aa = \lambda a^2 = \lambda \|a\|^2 \tag{8.17}$$

which is a scalar.

Linearly independent vectors

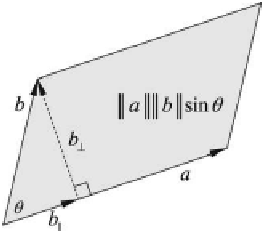

FIGURE 8.3.

With reference to Fig. 8.3

$$b = b_{\|} + b_{\perp}. \tag{8.18}$$

Therefore, we can write

$$ab = a(b_{\parallel} + b_{\perp}) \tag{8.19}$$

and

$$ab = ab_{\parallel} + ab_{\perp}. \tag{8.20}$$

Let's examine the RHS products of Eq. (8.20):
$ab_{\parallel}$: As a and $b_{\parallel}$ are linearly dependent, $ab_{\parallel}$ is a scalar. Furthermore,

$$ab_{\parallel} = \|a\|\|b\|\cos\theta = a \cdot b \tag{8.21}$$

which is defined as the inner product, or the inner product, and is symmetric.

$ab_\perp$: As a and $b_\perp$ are orthogonal

$$ab_\perp = \|a\|\|b\| \sin\theta = a \wedge b \quad (8.22)$$

which is defined as the outer product and is antisymmetric; i.e.

$$a \wedge b = -b \wedge a. \quad (8.23)$$

The area of the parallelogram formed by a and b in Fig. 8.20 is

$$\|a\|\|b\| \sin\theta. \quad (8.24)$$

Therefore,

$$\|a \wedge b\| = \|a\|\|b\| \sin\theta \quad (8.25)$$

which enables us to write Eq. (8.20) as

$$ab = a \cdot b + a \wedge b. \quad (8.26)$$

The parallel and orthogonal components created by $a \cdot b$ and $a \wedge b$ describe everything about the vectors a and b, which is why Clifford combined them into his geometric product. Furthermore, because these product components are linearly independent, the modulus of ab is computed using the Pythagorean rule:

$$\|ab\|^2 = \|a \cdot b\|^2 + \|a \wedge b\|^2$$

$$\|ab\|^2 = \|a\|^2\|b\|^2 \cos^2\theta + \|a\|^2\|b\|^2 \sin^2\theta$$

$$\|ab\|^2 = \|a\|^2\|b\|^2(\cos^2\theta + \sin^2\theta) \tag{8.27}$$

$$\|ab\| = \|a\|\|b\|. \tag{8.28}$$

Now we already know that $a \cdot b$ is a pure scalar and $a \wedge b$ is a directed area, which we suspect has an imaginary flavor. So it may seem strange adding two different mathematical objects together, but no stranger than a complex number. Nevertheless, we still require a name for this new object, which is a *multivector* and is described in section 8.5.

If we reverse the product to ba we have

$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b. \tag{8.29}$$

Note how the antisymmetry of the outer product introduces the negative sign.

Knowing that the geometric product is the sum of the inner and outer products, it is possible to define the inner and outer products in terms of the geometric product as follows.

Subtracting Eq. (8.29) from Eq. (8.26) we obtain

$$ab - ba = (a \cdot b + a \wedge b) - (a \cdot b - a \wedge b) = 2(a \wedge b) \tag{8.30}$$

therefore,

$$a \wedge b = \frac{1}{2}(ab - ba). \tag{8.31}$$

Similarly, adding Eq. (8.29) to Eq. (8.26) we obtain

$$ab + ba = 2a \cdot b \tag{8.32}$$

therefore,

$$a \cdot b = \frac{1}{2}(ab + ba). \tag{8.33}$$

These are important relationships and will be called upon frequently.

Now let's explore the geometric product further using the unit basis vectors for $\mathbb{R}^2$.

The product of identical basis vectors

Before we begin exploring this product, it is worth introducing a shorthand notation that simplifies our equations. Very often we have to write down a string of basis vectors such as $e_1e_2e_1$ which can also be written as e_{121}, and saves space on the printed page. In general this is expressed as:

$$e_ie_je_k \equiv e_{ijk}. \tag{8.34}$$

So let's start with the product e_1e_1:

$$e_1e_1 = e_1 \cdot e_1 + e_1 \wedge e_1. \tag{8.35}$$

Now we already know that $e_1 \wedge e_1 = 0$ and $e_1 \cdot e_1 = 1$, which means that

$$e_1e_1 = e_1^2 = 1. \tag{8.36}$$

Similarly,

$$e_2^2 = 1. \tag{8.37}$$

The product of orthogonal basis vectors

Next, the product e_1e_2:

$$e_1e_2 = e_1 \cdot e_2 + e_1 \wedge e_2. \tag{8.38}$$

Again, we know that $e_1 \cdot e_2 = 0$, which means that

$$e_1e_2 = e_1 \wedge e_2. \tag{8.39}$$

So, whenever we find the unit bivector $e_1 \wedge e_2$ we can substitute e_1e_2 or e_{12}.

Now let's compute the product e_2e_1:

$$e_2e_1 = e_2 \cdot e_1 + e_2 \wedge e_1 = e_2 \cdot e_1 - e_1 \wedge e_2. \tag{8.40}$$

But $e_2 \cdot e_1 = 0$, therefore,

$$e_2e_1 = -e_1 \wedge e_2 = -e_{12}. \tag{8.41}$$

The imaginary properties of the outer product

The imaginary properties of the outer product are revealed by evaluating the product $(e_1 \wedge e_2)^2$:

$$(e_1 \wedge e_2)^2 = (e_1 \wedge e_2)(e_1 \wedge e_2) = e_1e_2e_1e_2. \tag{8.42}$$

But as

$$e_2e_1 = -e_1e_2 \tag{8.43}$$

then

$$(e_1 \wedge e_2)^2 = -e_1e_1e_2e_2 = -e_1^2e_2^2. \tag{8.44}$$

But as

$$e_1^2 = e_2^2 = 1 \tag{8.45}$$

then

$$(e_1 \wedge e_2)^2 = -1. \tag{8.46}$$

So the unit bivector possess the same qualities as imaginary i in that it squares to -1.

Now this has all sorts of ramifications as it suggests that GA is related to complex numbers and possibly, quaternions, and could perform rotations in n-dimensions. At this point, the algebra explodes into many paths, which will have to be explored in turn.