GEOMETRIC ALGEBRA (CH. 7)

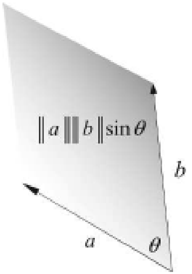
Aljabar Geometri (IF2123)

Sekolah Teknik Elektro dan Informatika
Institut Teknologi Bandung

Introduction: Length, area and volume

Primarily, GA manipulates vectors, although scalar quantities are easily integrated into the equations, but, for the moment, we will concentrate on the role vectors play within the algebra.

A single vector, independent of its spatial dimension, has two qualities: orientation and magnitude. Its orientation is determined by the sign of its components, whilst its magnitude is represented by its length, which in turn is derived from its components. A vector's orientation is reversed, simply by switching the signs of its components.



The product of two vectors can be used to represent the area of a parallelogram as shown in Fig. 7.1, where the area is given by

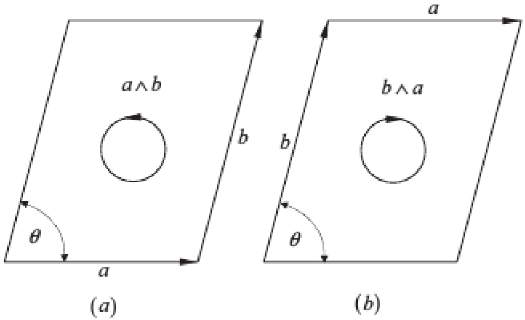
$$area = \|a\|\|b\| \sin\theta. \tag{7.1}$$

Because $\|a\|$ and $\|b\|$ are scalars, their order is immaterial. Furthermore, we have assumed that the angle θ is always positive, hence its sign is always positive, which is why area is normally regarded as a positive quantity.

Grassmann was aware that mathematics, especially determinants, supported positive *and* negative areas and volumes, and wanted to exploit this feature. His solution was to create a vector product that he called the *outer product* and written $a \wedge b$. The wedge symbol "$\wedge$" is why the product is also known as the *wedge product*, and it is worth noting that this symbol is also used by French mathematicians for the vector (cross) product. The outer product is sensitive to the order of the vectors it manipulates, and permits us to distinguish between $a \wedge b$ and $b \wedge a$. In fact, the algebra ensures that

$$a \wedge b = -b \wedge a. \tag{7.2}$$

Figure 7.2a shows that $a \wedge b$ creates an area from vectors a and b forming an anticlockwise rotation, whereas Fig. 7.2b shows that $b \wedge a$ creates an area from vectors b and a forming a clockwise rotation. The directed circle is included to remind us of the area's orientation.


a ∧ b
b
θ
a
(a)
a
b ∧ a
b
θ
(b)

The outer product

Now we already know that the magnitude of the vector product is given by

$$\|a \times b\| = \|a\|\|b\| \sin\theta \tag{7.3}$$

where θ is the angle between the two vectors. The outer product preserves this value but abandons the concept of a perpendicular vector. Instead, the value $\|a\|\|b\| \sin\theta$ is retained as the signed area formed by the two vectors.

Now although $\|a \wedge b\| = \|a\|\|b\| \sin\theta$, we must pose the question: What sort of object is $a \wedge b$? Well, for a start, it is not a vector, nor is it a simple scalar. In fact, we have to invent a new name, which is always unsettling as it is difficult to relate it to things with which we are familiar. Where the cross product $a \times b$ creates a vector, the outer product $a \wedge b$ is called a *bivector*, which is a totally new concept to grasp.

A bivector describes the orientation of a plane in terms of two vectors, and its magnitude is the area of the parallelogram formed by the vectors. Reversing the vector sequence in the product flips the sign of the area. The outer product has the same components as the cross product, but instead of using the components to form a vector, they become the projective characteristics of a planar surface.

Some algebreic properties

Even with our sketchy knowledge of a bivector, it is possible to describe how the outer product responds to parallel vectors. For example

$$\|a \wedge a\| = \|a\|\|a\| \sin 0^\circ = 0. \tag{7.4}$$

Although the outer product is antisymmetric, it behaves just like the scalar product when multiplying a group of vectors:

$$\text{scalar:} \quad a \cdot (b + c) = a \cdot b + a \cdot c \tag{7.5}$$

similarly

$$\text{outer:} \quad a \wedge (b + c) = a \wedge b + a \wedge c. \tag{7.6}$$

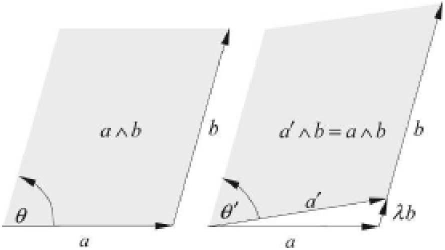
Visualizing the outer product

The cross product is easy to visualize: $a \times b = c$, where c is orthogonal to the plane containing a and b. The relative direction of c is determined by the right-hand rule where using one's right hand, where the thumb aligns with a, the first finger with b, and the middle finger aligns with c. The magnitude of c equals $\|a\|\|b\| \sin\theta$, where θ is the angle between a and b, and equals the area of the parallelogram formed by a and b. This relationship is shown in Fig. 7.3.

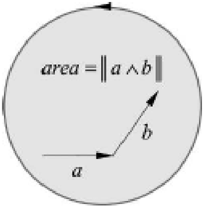
Visualizing the outer product is slightly different. It is true that the magnitude $\|a \wedge b\|$ is $\|a\|\|b\| \sin\theta$ which represents the area of the parallelogram formed by a and b, but consider what happens if we form the product $a' \wedge b$ where $a' = a + \lambda b$:

$$\begin{aligned} a' \wedge b &= (a + \lambda b) \wedge b \\ &= a \wedge b + \lambda b \wedge b \\ a' \wedge b &= a \wedge b. \end{aligned} \tag{7.7}$$

Two other vectors generate the same bivector! Figure 7.4 illustrates what is happening.



The area created by $a' \wedge b$ is identical to that created by $a \wedge b$, so there is no single parallelogram that represents $a \wedge b$ — there are an infinite number! So why bother trying to represent $a \wedge b$ as a parallelogram in the first place? Well, it was a starting point, but now that we have discovered this feature of the outer product, why not substitute another shape such as a circle instead of a parallelogram, and make the area of the circle equal to $\|a\|\|b\| \sin\theta$? That was a rhetorical question, but a useful suggestion, and Fig. 7.5 shows what is implied.



Orthogonal bases

If we continue with this notation the alphabet cannot support very high-dimensional spaces. An alternative convention is to use $e_1, e_2, e_3, \ldots e_n$ to represent the orthogonal unit basis vectors.

Using this notation we define two vectors in $\mathbb{R}^2$ as

$$a = a_1 e_1 + a_2 e_2 \tag{7.8}$$

$$b = b_1 e_1 + b_2 e_2. \tag{7.9}$$

We can now state the outer product as

$$a \wedge b = (a_1 e_1 + a_2 e_2) \wedge (b_1 e_1 + b_2 e_2) \tag{7.10}$$

which expands to

$$a \wedge b = a_1 b_1 (e_1 \wedge e_1) + a_1 b_2 (e_1 \wedge e_2) + a_2 b_1 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2). \tag{7.11}$$

Substituting the following observations

$$e_1 \wedge e_1 = e_2 \wedge e_2 = 0 \text{ and } e_2 \wedge e_1 = -e_1 \wedge e_2 \tag{7.12}$$

we obtain

$$a \wedge b = a_1 b_2 (e_1 \wedge e_2) - a_2 b_1 (e_1 \wedge e_2) \tag{7.13}$$

simplifying, we obtain

$$a \wedge b = (a_1 b_2 - a_2 b_1)(e_1 \wedge e_2). \tag{7.14}$$

The scalar term $a_1 b_2 - a_2 b_1$ in Eq. (7.14) looks familiar — in fact, it is the magnitude of the imaginary term of Eq. (3.17), the value of which equals $\|a\|\|b\| \sin\theta$, which is the area of the parallelogram formed by a and b. So in this context, the outer product $a \wedge b$ is a scalar area multiplying the unit bivector $e_1 \wedge e_2$, which just means that the area is associated with the plane defined by $e_1 \wedge e_2$. Figure 7.6 illustrates this relationship.

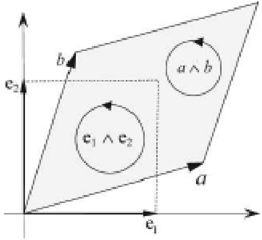


FIGURE 7.6.

Now let's compute $b \wedge a$:

$$b \wedge a = (b_1 e_1 + b_2 e_2) \wedge (a_1 e_1 + a_2 e_2)$$

which expands to

$$b \wedge a = a_1 b_1 (e_1 \wedge e_1) + a_2 b_1 (e_1 \wedge e_2) + a_1 b_2 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2). \tag{7.15}$$

Substituting the following observations

$$e_1 \wedge e_1 = e_2 \wedge e_2 = 0 \text{ and } e_2 \wedge e_1 = -e_1 \wedge e_2 \tag{7.16}$$

we obtain

$$b \wedge a = a_2 b_1 (e_1 \wedge e_2) - a_1 b_2 (e_1 \wedge e_2). \tag{7.17}$$

Simplifying, we obtain

$$b \wedge a = -(a_1 b_2 - a_2 b_1)(e_1 \wedge e_2) \tag{7.18}$$

which confirms that $b \wedge a = -a \wedge b$.

Now let's consider the outer product in $\mathbb{R}^3$:

$$a = a_1\mathrm{e}_1 + a_2\mathrm{e}_2 + a_3\mathrm{e}_3 \tag{7.19}$$

$$b = b_1\mathrm{e}_1 + b_2\mathrm{e}_2 + b_3\mathrm{e}_3. \tag{7.20}$$

The outer product is

$$a \wedge b = (a_1\mathrm{e}_1 + a_2\mathrm{e}_2 + a_3\mathrm{e}_3) \wedge (b_1\mathrm{e}_1 + b_2\mathrm{e}_2 + b_3\mathrm{e}_3) \tag{7.21}$$

which expands to

$$\begin{aligned} a \wedge b = {} & a_1b_1(\mathrm{e}_1 \wedge \mathrm{e}_1) + a_1b_2(\mathrm{e}_1 \wedge \mathrm{e}_2) + a_1b_3(\mathrm{e}_1 \wedge \mathrm{e}_3) + a_2b_1(\mathrm{e}_2 \wedge \mathrm{e}_1) + a_2b_2(\mathrm{e}_2 \wedge \mathrm{e}_2) \\ & + a_2b_3(\mathrm{e}_2 \wedge \mathrm{e}_3) + a_3b_1(\mathrm{e}_3 \wedge \mathrm{e}_1) + a_3b_2(\mathrm{e}_3 \wedge \mathrm{e}_2) + a_3b_3(\mathrm{e}_3 \wedge \mathrm{e}_3). \end{aligned} \tag{7.22}$$

Substituting

$$\mathrm{e}_1 \wedge \mathrm{e}_1 = \mathrm{e}_2 \wedge \mathrm{e}_2 = \mathrm{e}_3 \wedge \mathrm{e}_3 = 0 \tag{7.23}$$

and

$$\mathrm{e}_2 \wedge \mathrm{e}_1 = -\mathrm{e}_1 \wedge \mathrm{e}_2 \quad \mathrm{e}_1 \wedge \mathrm{e}_3 = -\mathrm{e}_3 \wedge \mathrm{e}_1 \quad \mathrm{e}_3 \wedge \mathrm{e}_2 = -\mathrm{e}_2 \wedge \mathrm{e}_3 \tag{7.24}$$

we obtain

$$a \wedge b = a_1 b_2 (e_1 \wedge e_2) - a_1 b_3 (e_3 \wedge e_1) - a_2 b_1 (e_1 \wedge e_2) \\ + a_2 b_3 (e_2 \wedge e_3) + a_3 b_1 (e_3 \wedge e_1) - a_3 b_2 (e_2 \wedge e_3). \tag{7.25}$$

Simplifying, we obtain

$$a \wedge b = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1. \tag{7.26}$$

You may be wondering why the unit basis bivectors in Eq. (7.26) have been chosen in this way, especially $e_3 \wedge e_1$. This could easily be $e_1 \wedge e_3$. To understand why, refer to Fig. 7.7, which shows a right-handed axial system and where each orthogonal plane is defined by its associated unit basis bivectors.

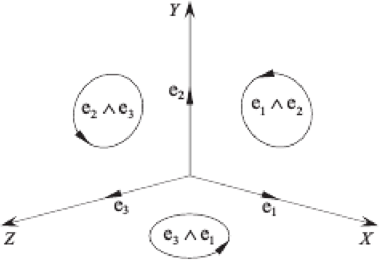


FIGURE 7.7.

Figure 7.7 also shows the orthogonal alignment of the Cartesian axes with the unit basis bivectors:

the x-axis is orthogonal to $e_2 \wedge e_3$

the y-axis is orthogonal to $e_3 \wedge e_1$

the z-axis is orthogonal to $e_1 \wedge e_2$

and if Eq. (7.26) is rearranged in this sequence we obtain

$$a \wedge b = (a_2b_3 - a_3b_2)\mathrm{e}_2 \wedge \mathrm{e}_3 + (a_3b_1 - a_1b_3)\mathrm{e}_3 \wedge \mathrm{e}_1 + (a_1b_2 - a_2b_1)\mathrm{e}_1 \wedge \mathrm{e}_2. \tag{7.27}$$

Now let's look at a definition of the cross product. We begin by declaring two vectors using the conventional orthogonal unit basis vectors i, j and k:

$$a = a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k} \tag{7.28}$$

$$b = b_1\mathrm{i} + b_2\mathrm{j} + b_3\mathrm{k}. \tag{7.29}$$

The cross product is

$$a \times b = (a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k}) \times (b_1\mathrm{i} + b_2\mathrm{j} + b_3\mathrm{k}) \tag{7.30}$$

which expands to

$$\begin{aligned} a \times b = {} & a_1b_1(\mathrm{i} \times \mathrm{i}) + a_1b_2(\mathrm{i} \times \mathrm{j}) + a_1b_3(\mathrm{i} \times \mathrm{k}) + a_2b_1(\mathrm{j} \times \mathrm{i}) + a_2b_2(\mathrm{j} \times \mathrm{j}) \\ & + a_2b_3(\mathrm{j} \times \mathrm{k}) + a_3b_1(\mathrm{k} \times \mathrm{i}) + a_3b_2(\mathrm{k} \times \mathrm{j}) + a_3b_3(\mathrm{k} \times \mathrm{k}). \end{aligned} \tag{7.31}$$

The magnitude of the cross product is $\|a\|\,\|b\| \sin\theta$, which means that

$$\mathrm{i} \times \mathrm{i} = \mathrm{j} \times \mathrm{j} = \mathrm{k} \times \mathrm{k} = 0. \tag{7.32}$$

Therefore,

$$a \times b = a_1b_2(\mathrm{i} \times \mathrm{j}) + a_1b_3(\mathrm{i} \times \mathrm{k}) + a_2b_1(\mathrm{j} \times \mathrm{i}) + a_2b_3(\mathrm{j} \times \mathrm{k}) + a_3b_1(\mathrm{k} \times \mathrm{i}) + a_3b_2(\mathrm{k} \times \mathrm{j}). \quad (7.33)$$

Because the cross product is antisymmetric

$$\mathrm{j} \times \mathrm{i} = -\mathrm{i} \times \mathrm{j} \quad \mathrm{k} \times \mathrm{j} = -\mathrm{j} \times \mathrm{k} \quad \mathrm{i} \times \mathrm{k} = -\mathrm{k} \times \mathrm{i}. \quad (7.34)$$

Substituting these relationships:

$$a \times b = a_1b_2(\mathrm{i} \times \mathrm{j}) - a_1b_3(\mathrm{k} \times \mathrm{i}) - a_2b_1(\mathrm{i} \times \mathrm{j}) + a_2b_3(\mathrm{j} \times \mathrm{k}) + a_3b_1(\mathrm{k} \times \mathrm{i}) - a_3b_2(\mathrm{j} \times \mathrm{k}). \quad (7.35)$$

Collecting up like terms:

$$a \times b = (a_2b_3 - a_3b_2)\mathrm{j} \times \mathrm{k} + (a_3b_1 - a_1b_3)\mathrm{k} \times \mathrm{i} + (a_1b_2 - a_2b_1)\mathrm{i} \times \mathrm{j}. \quad (7.36)$$

If we place Eqs. (7.27) and (7.36) together and substitute the e notation for i, j and k, we obtain

$$a \wedge b = (a_2b_3 - a_3b_2)\mathrm{e}_2 \wedge \mathrm{e}_3 + (a_3b_1 - a_1b_3)\mathrm{e}_3 \wedge \mathrm{e}_1 + (a_1b_2 - a_2b_1)\mathrm{e}_1 \wedge \mathrm{e}_2 \qquad (7.37)$$

$$a \times b = (a_2b_3 - a_3b_2)\mathrm{e}_2 \times \mathrm{e}_3 + (a_3b_1 - a_1b_3)\mathrm{e}_3 \times \mathrm{e}_1 + (a_1b_2 - a_2b_1)\mathrm{e}_1 \times \mathrm{e}_2. \qquad (7.38)$$

In the cross product, the terms $(a_2b_3 - a_3b_2)$, $(a_3b_1 - a_1b_3)$ and $(a_1b_2 - a_2b_1)$ are the components of an orthogonal vector, whereas in the outer product they become signed areas projected onto the planes defined by the unit bivectors $\mathrm{e}_2 \wedge \mathrm{e}_3$, $\mathrm{e}_3 \wedge \mathrm{e}_1$ and $\mathrm{e}_1 \wedge \mathrm{e}_2$. And in spite of there being such similarity between the two equations, it would be dangerous to conclude that $a \wedge b \equiv a \times b$.

What Hamilton had proposed was that

$$\mathrm{e}_2 \times \mathrm{e}_3 = \mathrm{e}_1 \quad \mathrm{e}_3 \times \mathrm{e}_1 = \mathrm{e}_2 \quad \mathrm{e}_1 \times \mathrm{e}_2 = \mathrm{e}_3 \qquad (7.39)$$

which is fine for $\mathbb{R}^3$, but is ambiguous for higher dimensions. So, in GA we substitute the outer product for the cross product and introduce the concept of a directed area, which holds for any number of dimensions.

We define two vectors as

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 \tag{7.40}$$

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3. \tag{7.41}$$

Starting with the plane containing e_1 and e_2, which is defined by $e_1 \wedge e_2$, the projections of a and b are a''' and b''', respectively, where

$$a''' = a_1 e_1 + a_2 e_2 \tag{7.42}$$

$$b''' = b_1 e_1 + b_2 e_2. \tag{7.43}$$

Therefore,

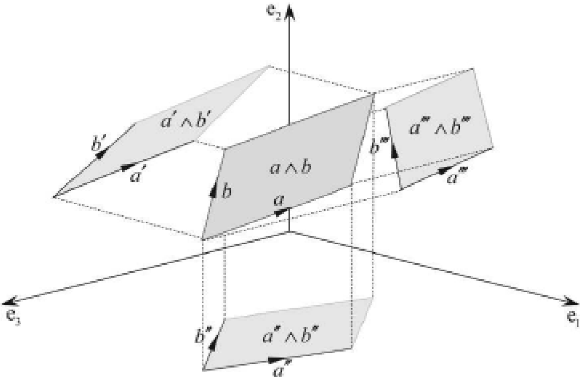
$$\begin{aligned} a''' \wedge b''' &= (a_1 e_1 + a_2 e_2) \wedge (b_1 e_1 + b_2 e_2) \\ &= a_1 b_1 (e_1 \wedge e_1) + a_1 b_2 (e_1 \wedge e_2) + a_2 b_1 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2) \\ a''' \wedge b''' &= (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 \end{aligned} \tag{7.44}$$

which is the last term in Eq. (7.27).

Similarly, we can show that

$$a' \wedge b' = (a_2 b_3 - a_3 b_2)\mathrm{e}_2 \wedge \mathrm{e}_3 \tag{7.45}$$

$$a'' \wedge b'' = (a_3 b_1 - a_1 b_3)\mathrm{e}_3 \wedge \mathrm{e}_1. \tag{7.46}$$



To illustrate this concept, consider two vectors a and b

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 \tag{7.47}$$

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3 \tag{7.48}$$

where

$$a_1 = 1 \quad a_2 = 0 \quad a_3 = 1$$

$$b_1 = 1 \quad b_2 = 1 \quad b_3 = 0 \tag{7.49}$$

which makes

$$a = e_1 + e_3 \quad b = e_1 + e_2. \tag{7.50}$$

Using Eq. (7.26)

$$a \wedge b = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1$$

$$a \wedge b = (1) e_1 \wedge e_2 + (-1) e_2 \wedge e_3 + (1) e_3 \wedge e_1. \tag{7.51}$$

The signed area on the plane $e_1 \wedge e_2$ is $+1$ and is shown in Fig. 7.9. The projected area is shown crosshatched.

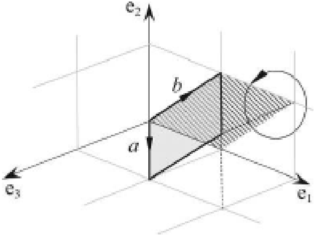


FIGURE 7.9.

Similarly, the signed area on the plane $e_2 \wedge e_3$ is -1 and is shown in Fig. 7.10. Note that the direction of the projected area opposes the direction of $e_2 \wedge e_3$.

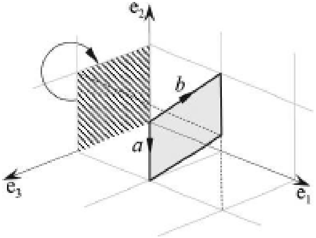


FIGURE 7.10.

And the signed area on the plane $e_3 \wedge e_1$ is $+1$, and is shown in Fig. 7.11.

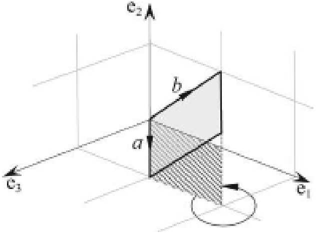


FIGURE 7.11.

Now let's compute the magnitude of the bivector $a \wedge b$.

To begin with, we need to know the angle between a and b, which is revealed using the dot product:

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\|a\|\|b\|}\right)$$
$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}\sqrt{2}}\right) = 60^\circ. \tag{7.52}$$

Therefore,

$$\|a \wedge b\| = \|a\|\|b\| \sin 60^\circ$$
$$\|a \wedge b\| = \sqrt{2}\sqrt{2}\frac{\sqrt{3}}{2} = \sqrt{3}. \tag{7.53}$$

The next question to pose is whether this value is related to the other three areas? Well the answer is "yes", and for a very good reason:

$$\|a \wedge b\|^2 = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 \tag{7.54}$$

therefore,

$$\sqrt{3}^2 = (1)^2 + (-1)^2 + (1)^2 = 3. \tag{7.55}$$

Remember, that the cross product uses these coefficients as Cartesian components of the axial vector and satisfy the Pythagorean rule:

$$\|a\|^2 = a_1^2 + a_2^2 + a_3^2. \tag{7.56}$$

To prove that this holds, we need to show that Eq. (7.54) is correct.

Expanding the LHS of Eq. (7.54):

$$\|a \wedge b\|^2 = \|a\|^2\|b\|^2 \sin^2\theta = \|a\|^2\|b\|^2(1 - \cos^2\theta)$$

$$\|a \wedge b\|^2 = \|a\|^2\|b\|^2 - \|a\|^2\|b\|^2 \cos^2\theta. \tag{7.57}$$

From the dot product

$$\cos^2\theta = \frac{(a_1b_1 + a_2b_2 + a_3b_3)^2}{\|a\|^2\|b\|^2}. \tag{7.58}$$

Therefore,

$$\|a \wedge b\|^2 = \|a\|^2\|b\|^2 - (a_1b_1 - a_2b_2 - a_3b_3)^2$$

$$\|a \wedge b\|^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 - a_2b_2 - a_3b_3)^2$$

Therefore,

$$\|a \wedge b\|^2 = \|a\|^2 \|b\|^2 - (a_1b_1 - a_2b_2 - a_3b_3)^2$$
$$\|a \wedge b\|^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 - a_2b_2 - a_3b_3)^2$$

and we obtain

$$\begin{aligned}\|a \wedge b\|^2 &= (a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2) + (a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2) \\ &\quad + (a_3^2b_1^2 - 2a_3a_1b_3b_1 + a_1^2b_3^2) \\ \|a \wedge b\|^2 &= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2. \end{aligned} \tag{7.59}$$

Therefore, Eq. (7.54) is correct.

Now, as

$$\|a \wedge b\| = \|a\| \|b\| \sin\theta \tag{7.60}$$

$$\|a \wedge b\|^2 = \|a\|^2 \|b\|^2 \sin^2\theta \tag{7.61}$$

and

$$\|a\|\|b\| \sin^2\theta = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 \tag{7.62}$$

therefore

$$\theta = \sin^{-1}\left(\frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2}}{\|a\|\|b\|}\right). \tag{7.63}$$

Substituting the values for the above example:

$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}\sqrt{2}}\right) = 60°. \tag{7.64}$$

The beauty of the outer product is that it works in any number of dimensions. For example, we can create two vectors in $\mathbb{R}^4$ as follows:

$$a = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_4\mathbf{e}_4 \tag{7.65}$$

$$b = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_4\mathbf{e}_4 \tag{7.66}$$

and form their outer product:

$$a \wedge b = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_4\mathbf{e}_4) \wedge (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_4\mathbf{e}_4). \tag{7.67}$$

This explodes into

$$
\begin{aligned}
a \wedge b = a_1 b_1(e_1 \wedge e_1) + a_1 b_2(e_1 \wedge e_2) + a_1 b_3(e_1 \wedge e_3) + a_1 b_4(e_1 \wedge e_4) \\
+ a_2 b_1(e_2 \wedge e_1) + a_2 b_2(e_2 \wedge e_2) + a_2 b_3(e_2 \wedge e_3) + a_2 b_4(e_2 \wedge e_4) \\
+ a_3 b_1(e_3 \wedge e_1) + a_3 b_2(e_3 \wedge e_2) + a_3 b_3(e_3 \wedge e_3) + a_3 b_4(e_3 \wedge e_4) \\
+ a_4 b_1(e_4 \wedge e_1) + a_4 b_2(e_4 \wedge e_2) + a_4 b_3(e_4 \wedge e_3) + a_4 b_4(e_4 \wedge e_4)
\end{aligned}
$$

and collapses to

$$
\begin{aligned}
a \wedge b = (a_1 b_2 - a_2 b_1)(e_1 \wedge e_2) + (a_2 b_3 - a_3 b_2)(e_2 \wedge e_3) + (a_3 b_1 - a_1 b_3)(e_3 \wedge e_1) \\
+ (a_1 b_4 - a_4 b_1)(e_1 \wedge e_4) + (a_2 b_4 - a_4 b_2)(e_2 \wedge e_4) + (a_3 b_4 - a_4 b_3)(e_3 \wedge e_4) \qquad (7.68)
\end{aligned}
$$

which resolves the outer product into six bivectors.

As a final example, let's consider two vectors in $\mathbb{R}^4$ and compute their outer product. The vectors are

$$a = e_1 + e_3 + e_4 \tag{7.71}$$

$$b = e_1 + e_2 + e_4. \tag{7.72}$$

Then

$$\|a\| = \sqrt{3} \quad \|b\| = \sqrt{3} \tag{7.73}$$

and the separating angle θ is

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \simeq 48.19^\circ. \tag{7.74}$$

Similarly,

$$\theta = \sin^{-1}\left(\frac{\sqrt{5}}{3}\right) \simeq 48.19^\circ. \tag{7.75}$$

Substituting the vectors into Eq. (7.68):

$$a \wedge b = (1)(e_1 \wedge e_2) + (-1)(e_2 \wedge e_3) + (1)(e_3 \wedge e_1) + (-1)(e_2 \wedge e_4) + (1)(e_3 \wedge e_4). \qquad (7.76)$$

Therefore, $\|a \wedge b\|$ is given by

$$\|a \wedge b\| = \|a\|\|b\| \sin\theta = \sqrt{3}\sqrt{3}\sin 48.19^\circ \simeq 2.2361. \qquad (7.77)$$

Finally, let's show that the $\mathbb{R}^4$ equivalent of Eq. (7.54) still holds:

$$\begin{aligned}\|a \wedge b\|^2 &= |a_1b_2 - a_2b_1|^2 + |a_2b_3 - a_3b_2|^2 + |a_3b_1 - a_1b_3|^2 \\ &\quad + |a_1b_4 - a_4b_1|^2 + |a_2b_4 - a_4b_2|^2 + |a_3b_4 - a_4b_3|^2 \\ 2.2361^2 &= (1)^2 + (-1)^2 + (1)^2 + (0)^2 + (-1)^2 + (1)^2 = 5. \qquad (7.78)\end{aligned}$$

Area of a triangle

There are many ways to find the area of a triangle, but the one proposed here uses a triangle's vertex coordinates, as shown in Fig. 7.12a. The triangle has vertices A, B, C defined in an anticlockwise order, and its area is given by

$$area = \frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}. \tag{7.79}$$

Using the coordinates from Fig. 7.12a we have

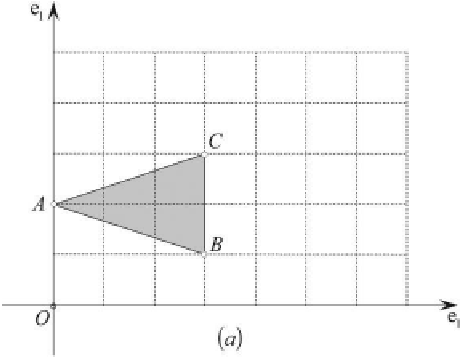
$$area = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix} \tag{7.80}$$

$$area = \frac{1}{2}(9 + 6 - 6 - 3) = +3 \tag{7.81}$$

which is correct. Note that reversing the triangle's vertex sequence creates a negative area:

$$area = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 3 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$area = \frac{1}{2}(3 + 6 - 6 - 9) = -3. \tag{7.82}$$


e1
C
A
B
O
e1
(a)

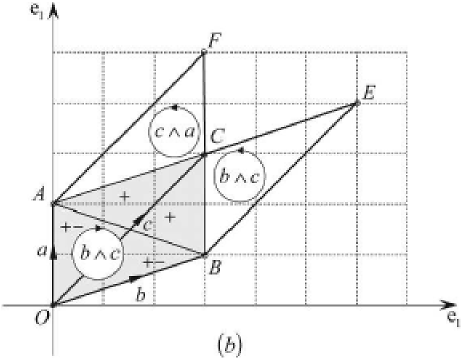


FIGURE 7.12.

The sum of the outer products become

$$area\ \Delta ABC = \frac{1}{2}[(a \wedge b) + (b \wedge c) + (c \wedge a)] \tag{7.83}$$

which expand to

$$area\ \Delta ABC = \frac{1}{2}(x_A y_B - y_A x_B + x_B y_C - y_B x_C + x_C y_A - y_C x_A)$$

and

$$area\ \Delta ABC = \frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}. \tag{7.84}$$

What is useful about summing these outer products is that it works for any irregular shape.

The sine rule

$$\frac{H}{B} = \sin\alpha \quad \text{and} \quad \frac{H}{A} = \sin\beta \tag{7.85}$$

from which we can write

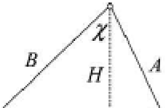
$$B\sin\alpha = A\sin\beta \tag{7.86}$$

or

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta}. \tag{7.87}$$

Using another vertex and an associated perpendicular we can show that

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\chi}. \tag{7.88}$$



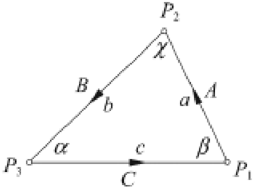


FIGURE 7.14.

The GA approach is to remember that the outer product includes a sine function and computes an area. Therefore, we develop Fig. 7.13 to include three vectors as shown in Fig. 7.14 where

$$A = \|a\| \quad B = \|b\| \quad C = \|c\| \tag{7.89}$$

To find λ we eliminate ε by multiplying Eq. (7.98) by y_b and Eq. (7.99) by x_b:

$$x_r y_b + \lambda x_a y_b = x_s y_b + \varepsilon x_b y_b \tag{7.100}$$

$$x_b y_r + \lambda x_b y_a = x_b y_s + \varepsilon x_b y_b. \tag{7.101}$$

Subtracting Eq. (7.101) from Eq. (7.100):

$$x_r y_b - x_b y_r + \lambda(x_a y_b - x_b y_a) = x_s y_b - x_b y_s \tag{7.102}$$

where

$$\lambda = \frac{x_b(y_r - y_s) - y_b(x_r - x_s)}{x_a y_b - x_b y_a}. \tag{7.103}$$

Intersection of two lines

The traditional way of calculating the intersection point of two lines in a plane is to define two vectors as shown in Fig. 7.15, where

$$p = r + \lambda a \quad \lambda \in \mathbb{R} \tag{7.95}$$

$$p = s + \varepsilon b \quad \varepsilon \in \mathbb{R}. \tag{7.96}$$

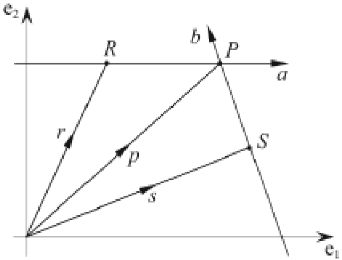


FIGURE 7.15.

Therefore,

$$r + \lambda a = s + \varepsilon b. \tag{7.97}$$

From Eq. (7.95) we can write

$$x_r + \lambda x_a = x_s + \varepsilon x_b \tag{7.98}$$

$$y_r + \lambda y_a = y_s + \varepsilon y_b. \tag{7.99}$$

To find λ we eliminate ε by multiplying Eq. (7.98) by y_b and Eq. (7.99) by x_b:

$$x_r y_b + \lambda x_a y_b = x_s y_b + \varepsilon x_b y_b \tag{7.100}$$

$$x_b y_r + \lambda x_b y_a = x_b y_s + \varepsilon x_b y_b. \tag{7.101}$$

Subtracting Eq. (7.101) from Eq. (7.100):

$$x_r y_b - x_b y_r + \lambda(x_a y_b - x_b y_a) = x_s y_b - x_b y_s \tag{7.102}$$

where

$$\lambda = \frac{x_b(y_r - y_s) - y_b(x_r - x_s)}{x_a y_b - x_b y_a}. \tag{7.103}$$

Let's test this with the following vectors

$$r = \mathrm{j} \quad a = 2\mathrm{i} - \mathrm{j} \tag{7.104}$$

$$s = 2\mathrm{j} \quad b = 2\mathrm{i} - 2\mathrm{j} \tag{7.105}$$

$$\lambda = \frac{2(1-2) + 2(0-0)}{-4+2} = \frac{-2}{-2} = 1 \tag{7.106}$$

therefore,

$$p = \mathrm{j} + 2\mathrm{i} - \mathrm{j} = 2\mathrm{i} \tag{7.107}$$

and the point of intersection is $(2, 0)$.

Another approach is to reason that

$$p = \alpha a + \beta b \tag{7.108}$$

therefore, we can write

$$x_p = \alpha x_a + \beta x_b \tag{7.109}$$

$$y_p = \alpha y_a + \beta y_b. \tag{7.110}$$

To find α we eliminate β by multiplying Eq. (7.109) by y_b and Eq. (7.110) by x_b:

$$x_p y_b = \alpha x_a y_b + \beta x_b y_b \tag{7.111}$$

$$x_b y_p = \alpha x_b y_a + \beta x_b y_b. \tag{7.112}$$

Subtracting Eq. (7.112) from Eq. (7.111) we obtain

$$x_p y_b - x_b y_p = \alpha x_a y_b - \alpha x_b y_a = \alpha (x_a y_b - x_b y_a) \tag{7.113}$$

where

$$\alpha = \frac{x_p y_b - x_b y_p}{x_a y_b - x_b y_a} = \frac{\begin{vmatrix} x_p & y_p \\ x_b & y_b \end{vmatrix}}{\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}}. \tag{7.114}$$

To find β we eliminate α by multiplying Eq. (7.109) by y_a and Eq. (7.110) by x_a:

$$x_p y_a = \alpha x_a y_a + \beta x_b y_a \tag{7.115}$$

$$x_a y_p = \alpha x_a y_a + \beta x_a y_b. \tag{7.116}$$

Subtracting Eq. (7.116) from Eq. (7.115) we obtain

$$x_p y_a - x_a y_p = \beta x_b y_a - \beta x_a y_b = \beta (x_b y_a - x_a y_b) \tag{7.117}$$

where

$$\beta = \frac{x_p y_a - x_a y_p}{x_b y_a - x_a y_b} = \frac{\begin{vmatrix} x_p & y_p \\ x_a & y_a \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}}. \tag{7.118}$$

Using Eq. (7.114) and Eq. (7.118) we can rewrite Eq. (7.108) as

$$p = \frac{\begin{vmatrix} x_p & y_p \\ x_b & y_b \end{vmatrix}}{\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}} a + \frac{\begin{vmatrix} x_p & y_p \\ x_a & y_a \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}} b. \tag{7.119}$$

The problem with Eq. (7.119) is that the determinants reference the coordinates of the point we are trying to discover. Nevertheless, let's continue and write Eq. (7.119) using outer products

$$p = \frac{p \wedge b}{a \wedge b} a + \frac{p \wedge a}{b \wedge a} b. \tag{7.120}$$

Figure 7.16a provides a graphical interpretation of part of Eq. (7.120) where the parallelogram formed by the outer product $p \wedge a$ is identical to the outer product formed by $r \wedge a$. Which means that we can substitute $r \wedge a$ for $p \wedge a$ in Eq. (7.120):

$$p = \frac{p \wedge b}{a \wedge b} a + \frac{r \wedge a}{b \wedge a} b. \tag{7.121}$$

Similarly, in Fig. 7.16b the parallelogram formed by the outer product $p \wedge b$ is identical to the outer product formed by $s \wedge b$. Which means that we can substitute $s \wedge b$ for $p \wedge b$ in Eq. (7.121):

$$p = \frac{s \wedge b}{a \wedge b}a + \frac{r \wedge a}{b \wedge a}b. \tag{7.122}$$

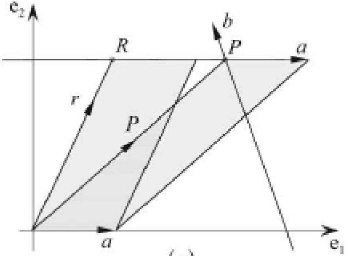
The positions of R and S are not very important, as they could be anywhere along the two vectors, even positioned as shown in Fig. 7.17:

In Fig. 7.17 the three parallelograms: $OSTU$, $OVWR$ and $OVXU$ have areas:

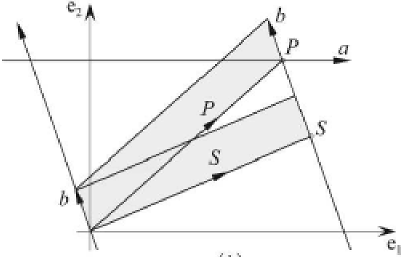
$$area\ OSTU = s \wedge b \tag{7.123}$$

$$area\ OVWR = r \wedge a \tag{7.124}$$

$$area\ OVXU = a \wedge b. \tag{7.125}$$



(a)



(b)

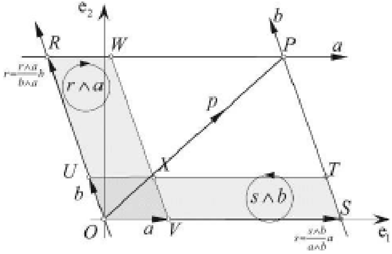


Figure 7.17.

Simply by relocating S and R, we have created a convenient visual symmetry where

$$s = \frac{s \wedge b}{a \wedge b} a \tag{7.126}$$

and

$$r = \frac{r \wedge a}{b \wedge a} b. \tag{7.127}$$

Note how $s \wedge b$ and $a \wedge b$ are in the same sense, whilst $r \wedge a$ and $b \wedge a$ are in the opposite sense. Observe, also, from Fig. (7.17) why

$$\frac{s}{a} = \frac{s \wedge b}{a \wedge b} \tag{7.128}$$

and

$$\frac{r}{b} = \frac{r \wedge a}{b \wedge a}. \tag{7.129}$$

It now becomes obvious that

$$p = s + r = \frac{s \wedge b}{a \wedge b}a + \frac{r \wedge a}{b \wedge a}b \tag{7.130}$$

where the solution to the problem is based upon the ratios of areas of parallelograms!

Let's test Eq. (7.130) using the same vectors above:

$$r = e_2 \qquad a = 2e_1 - e_2 \tag{7.131}$$

$$s = 2e_2 \quad b = 2e_1 - 2e_2 \tag{7.132}$$

$$p = \frac{(2e_2) \wedge (2e_1 - 2e_2)}{(2e_1 - e_2) \wedge (2e_1 - 2e_2)}(2e_1 - e_2) + \frac{e_2 \wedge (2e_1 - e_2)}{(2e_1 - 2e_2) \wedge (2e_1 - e_2)}(2e_1 - 2e_2)$$

$$p = \frac{-4(e_1 \wedge e_2)}{-4(e_1 \wedge e_2) + 2(e_1 \wedge e_2)}(2e_1 - e_2) + \frac{-2(e_1 \wedge e_2)}{-2(e_1 \wedge e_2) + 4(e_1 \wedge e_2)}(2e_1 - 2e_2)$$

$$p = 2(2e_1 - e_2) - (2e_1 - 2e_2) = 2e_1. \tag{7.133}$$

Therefore, the point of intersection is $(2, 0)$. Which is the same as the previous result.

We have spent some time exploring the above techniques, which in some cases are quite tedious. However, the conformal model, which is explored in chapter 11, simplifies the whole process.