MAC 2103

Module 12

Eigenvalues and Eigenvectors

Learning Objectives

Upon completing this module, you should be able to:

1. Solve the eigenvalue problem by finding the eigenvalues and the corresponding eigenvectors of an n x n matrix. Find the algebraic multiplicity and the geometric multiplicity of an eigenvalue.
2. Find a basis for each eigenspace of an eigenvalue.
3. Determine whether a matrix A is diagonalizable.
4. Find a matrix P, P^{-1}, and D that diagonalize A if A is diagonalizable.
5. Find an orthogonal matrix P with $P^{-1} = P^T$ and D that diagonalize A if A is symmetric and diagonalizable.
6. Determine the power and the eigenvalues of a matrix, A^k.

Eigenvalues and Eigenvectors

The major topics in this module:

Eigenvalues, Eigenvectors, Eigenspace, Diagonalization and Orthogonal Diagonalization

What are Eigenvalues and Eigenvectors?

If A is an n x n matrix and λ is a scalar for which A**x** = λ**x** has a nontrivial solution **x** $\in \Re^n$, then λ is an eigenvalue of A and **x** is a corresponding eigenvector of A. A**x** = λ**x** is called the eigenvalue problem for A.

Note that we can rewrite the equation A**x** = λ**x** = λI_n **x** as follows:

$\lambda I_n \mathbf{x} - A\mathbf{x} = \mathbf{0}$ or $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$. $\mathbf{x} = \mathbf{0}$ is the trivial solution. But our solutions must be nonzero vectors called eigenvectors that correspond to each of the distinct eigenvalues.

What are Eigenvalues and Eigenvectors?

Since we seek a nontrivial solution to $(\lambda I_n - A)\mathbf{x} = (\lambda I - A)\mathbf{x} = \mathbf{0}$, $\lambda I - A$ must be singular to have solutions $\mathbf{x} \neq \mathbf{0}$. This means that the $\det(\lambda I - A) = 0$.

The $\det(\lambda I - A) = p(\lambda) = 0$ is the characteristic equation, where $\det(\lambda I - A) = p(\lambda)$ is the characteristic polynomial. The $\deg(p(\lambda)) = n$ and the n roots of $p(\lambda)$, $\lambda_1, \lambda_2, \ldots, \lambda_n$, are the eigenvalues of A. The polynomial $p(\lambda)$ always has n roots, so the zeros always exist; but some may be complex and some may be repeated. In our examples, all of the roots will be real.

For each λ_i we solve for $\mathbf{x_i} = \mathbf{p_i}$ the corresponding eigenvector, and $A\mathbf{p_i} = \lambda_i \mathbf{p_i}$ for each distinct eigenvalue.

How to Solve the Eigenvalue Problem, A**x** = λ**x**?

Example 1: Find the eigenvalues and the corresponding eigenvectors of A.

$$A = \begin{bmatrix} -3 & -2 \\ -5 & 0 \end{bmatrix}$$

Step 1: Find the characteristic equation of A and solve for its eigenvalues.

$$p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda + 3 & -2 \\ -5 & \lambda \end{vmatrix} = 0$$

$$= (\lambda + 3)\lambda - (-2)(-5) = \lambda^2 + 3\lambda - 10 = (\lambda + 5)^1(\lambda - 2)^1 = 0$$

$$\textit{Thus, the eigenvalues are } \lambda_1 = -5,\ \lambda_2 = 2.$$

Each eigenvalue has algebraic multiplicity 1.

$$\textit{Let } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} \textit{ which is a diagonal matrix.}$$

How to Solve the Eigenvalue Problem, A**x** = λ**x**?(Cont.)

Step 2: Use Gaussian elimination with back-substitution to solve the $(\lambda I - A)\,\mathbf{x} = \mathbf{0}$ for λ_1 and λ_2.

The augmented matrix for the system with $\lambda_1 = -5$:

$$\left[-5I - A \mid \vec{0}\right] = \begin{bmatrix} -2 & 2 & 0 \\ 5 & -5 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} -\frac{1}{2}r1 \to r1 \\ r2 \end{matrix} \begin{bmatrix} 1 & -1 & 0 \\ 5 & -5 & 0 \end{bmatrix} \; : \; \begin{matrix} r1 \\ -5r1 + r2 \to r2 \end{matrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The second column is not a leading column, so $x_2 = t$ is a free variable, and $x_1 = x_2 = t$. Thus, the solution corresponding to $\lambda_1 = -5$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, t \neq 0.$$

How to Solve the Eigenvalue Problem, A**x** = λ**x**? (Cont.)

Since t is a free variable, there are infinitely many eigenvectors. For convenience, we choose t = 1, and

$$\vec{x} = \vec{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

as the eigenvector for λ_1 = -5. If we want $\mathbf{p_1}$ to be a unit vector, we will choose t so that

$$\vec{p}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

However, t =1 is fine in this problem.

The augmented matrix for the system with λ_2 = 2:

$$\left[2I - A \,|\, \vec{0}\right] = \begin{bmatrix} 5 & 2 & 0 \\ 5 & 2 & 0 \end{bmatrix}$$

$$: \begin{array}{c} \frac{1}{5}r1 \to r1 \\ r2 \end{array} \begin{bmatrix} 1 & \frac{2}{5} & 0 \\ 5 & 2 & 0 \end{bmatrix} \quad : \begin{array}{c} r1 \\ -5r1 + r2 \to r2 \end{array} \begin{bmatrix} 1 & \frac{2}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

How to Solve the Eigenvalue Problem, A**x** = λ**x**? (Cont.)

Again, the second column is not a leading column, so x_2 = t is a free variable, and $x_1 = -2x_2 / 5 = -2t / 5$. Thus, the solution corresponding to $\lambda_1 = 2$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}, t \neq 0.$$

For convenience, we choose t = 5 and

$$\vec{x} = \vec{p}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

as the eigenvector for $\lambda_1 = 2$. Alright, we have finished solving the eigenvalue problem for

$$A = \begin{bmatrix} -3 & -2 \\ -5 & 0 \end{bmatrix}.$$

Eigenspaces

For each eigenvalue λ, there is an eigenspace E_λ with a basis formed from the linearly independent eigenvectors for λ. The $\dim(E_\lambda)$ is the geometric multiplicity of λ, which is the number of linearly independent eigenvectors associated with λ. We will see that the geometric multiplicity equals the algebraic multiplicity for each eigenvalue.

$$B_1 = \{\vec{p}_1\} \textit{ is a basis for } E_{\lambda_1}, \textit{ the eigenspace of } \lambda_1,$$

$$\textit{and } B_2 = \{\vec{p}_2\} \textit{ is a basis for } E_{\lambda_2}, \textit{ the eigenspace of } \lambda_2.$$

$$So,\ E_{\lambda_1} = span(B_1) = span(\{\vec{p}_1\}) = span\left(\begin{bmatrix} 1 & 1 \end{bmatrix}^T\right),$$

$$E_{\lambda_2} = span(B_2) = span(\{\vec{p}_2\}) = span\left(\begin{bmatrix} -2 & 5 \end{bmatrix}^T\right)$$

$$\textit{and } \dim(E_{\lambda_1}) = \dim(E_{\lambda_2}) = 1.$$

How to Determine if a Matrix A is Diagonalizable?

If P = [$\mathbf{p}_1$ $\mathbf{p}_2$], then AP = PD even if A is not diagonalizable.

$$\text{Since } AP = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ -5 & 0 \end{bmatrix} = A\begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix} = \begin{bmatrix} A\vec{p}_1 & A\vec{p}_2 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -4 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} -5(1) & 2(-2) \\ -5(1) & 2(5) \end{bmatrix} = \begin{bmatrix} -5\begin{bmatrix} 1 \\ 1 \end{bmatrix} & 2\begin{bmatrix} -2 \\ 5 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \vec{p}_1 & \lambda_2 \vec{p}_2 \end{bmatrix} = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -2 \\ 5 \end{bmatrix} \end{bmatrix}\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} = PD.$$

How to Determine if a Matrix A is Diagonalizable?

So, our two distinct eigenvalues both have algebraic multiplicity 1 and geometric multiplicity 1. This ensures that $\mathbf{p_1}$ and $\mathbf{p_2}$ are not scalar multiples of each other; thus, $\mathbf{p_1}$ and $\mathbf{p_2}$ are linearly independent eigenvectors of A.

Since A is 2 x 2 and there are two linearly independent eigenvectors from the solution of the eigenvalue problem, A is diagonalizable and $P^{-1}AP = D$.

We can now construct P, P^{-1} and D. Let

$$P = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -2 \\ 5 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}.$$

$$Then,\ P^{-1} = \begin{bmatrix} 5/7 & 2/7 \\ -1/7 & 1/7 \end{bmatrix} \ and\ D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}.$$

How to Determine if a Matrix A is Diagonalizable? (Cont.)

Note that, if we multiply both sides on the left by P^{-1}, *then*

$$AP = A\begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix} == \begin{bmatrix} A\vec{p}_1 & A\vec{p}_2 \end{bmatrix} = \begin{bmatrix} -5 & -4 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} -5(1) & 2(-2) \\ -5(1) & 2(5) \end{bmatrix}$$

$$= \begin{bmatrix} -5\begin{bmatrix} 1 \\ 1 \end{bmatrix} & 2\begin{bmatrix} -2 \\ 5 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{p}_1 & \lambda_2\vec{p}_2 \end{bmatrix} = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -2 \\ 5 \end{bmatrix} \end{bmatrix}\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} = PD \quad \textit{becomes}$$

$$P^{-1}AP = \begin{bmatrix} 5/7 & 2/7 \\ -1/7 & 1/7 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5/7 & 2/7 \\ -1/7 & 1/7 \end{bmatrix}\begin{bmatrix} -5 & -4 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} -35/7 & 0/7 \\ 0/7 & 14/7 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} = D.$$

How to Determine if a Matrix A is Diagonalizable? (Cont.)

Example 2: Find the eigenvalues and eigenvectors for A.

$$A = \begin{bmatrix} 3 & -4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 1: Find the eigenvalues for A.

Recall: The determinant of a triangular matrix is the product of the elements at the diagonal. Thus, the characteristic equation of A is

$$p(\lambda) = \det(\lambda I - A) = |\lambda I - A| = 0$$

$$= \begin{vmatrix} \lambda - 3 & 4 & 0 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 3)^2 (\lambda - 1)^1 = 0.$$

$\lambda_1 = 1$ has algebraic multiplicity 1 and $\lambda_2 = 3$ has algebraic multiplicity 2.

How to Determine if a Matrix A is Diagonalizable? (Cont.)

Step 2: Use Gaussian elimination with back-substitution to solve (λI - A) **x** = **0**. For $\lambda_1 = 1$, the augmented matrix for the system is

$$[I - A \mid \vec{0}] = \begin{bmatrix} -2 & 4 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{array}{c} -\tfrac{1}{2} r1 \to r1 \\ r2 \\ r3 \end{array} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$: \begin{array}{c} r1 \\ -\tfrac{1}{2} r2 \to r2 \\ r3 \end{array} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Column 3 is not a leading column and $x_3 = t$ is a free variable. The geometric multiplicity of $\lambda_1 = 1$ is one, since there is only one free variable. $x_2 = 0$ and $x_1 = 2x_2 = 0$.

How to Determine if a Matrix A is Diagonalizable? (Cont.)

The eigenvector corresponding to $\lambda_1 = 1$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.\ \textit{If we choose } t = 1,\ \textit{then } \vec{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\ \textit{is}$$

our choice for the eigenvector.

$B_1 = \{\vec{p}_1\}$ *is a basis for the eigenspace,* E_{λ_1}, *with* $\dim(E_{\lambda_1}) = 1.$

The dimension of the eigenspace is 1 because the eigenvalue has only one linearly independent eigenvector. Thus, the geometric multiplicity is 1 and the algebraic multiplicity is 1 for $\lambda_1 = 1$.

How to Determine if a Matrix A is Diagonalizable? (Cont.)

The augmented matrix for the system with $\lambda_2 = 3$ is

$$\left[3I - A \,|\, \vec{0}\right] = \begin{bmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} : \begin{array}{c} \frac{1}{4}r1 \to r1 \\ r2 \\ r3 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$: \begin{array}{c} r1 \\ r3 \to r2 \\ r2 \to r3 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{array}{c} r1 \\ \frac{1}{2}r2 \to r2 \\ r3 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Column 1 is not a leading column and $x_1 = t$ is a free variable. Since there is only one free variable, the geometric multiplicity of λ_2 is one.

How to Determine if a Matrix A is Diagonalizable? (Cont.)

$x_2 = x_3 = 0$ and the eigenvector corresponding to $\lambda_2 = 3$ is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \textit{we choose } t = 1, \textit{ and } \vec{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \textit{ is}$$

our choice for the eigenvector.

$$B_2 = \{\vec{p}_2\} \textit{ is a basis for the eigenspace, } E_{\lambda_2}, \textit{ with } \dim(E_{\lambda_2}) = 1.$$

The dimension of the eigenspace is 1 because the eigenvalue has only one linearly independent eigenvector. Thus, the geometric multiplicity is 1 while the algebraic multiplicity is 2 for $\lambda_2 = 3$. This means there will not be enough linearly independent eigenvectors for A to be diagonalizable. Thus, A is not diagonalizable whenever the geometric multiplicity is less than the algebraic multiplicity for any eigenvalue.

How to Determine if a Matrix A is Diagonalizable? (Cont.)

$$\text{This time, } AP = A\begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_2 \end{bmatrix} = \begin{bmatrix} A\vec{p}_1 & A\vec{p}_2 & A\vec{p}_2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1\vec{p}_1 & \lambda_2\vec{p}_2 & \lambda_2\vec{p}_2 \end{bmatrix} = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = PD.$$

P^{-1} *does not exist since the columns of* P *are not linearly independent.*

It is not possible to solve for $D = P^{-1}AP$, *so* A *is not diagonalizable.*

A is Diagonalizable iff A is Similar to a Diagonal Matrix

For A, an n x n matrix, with characteristic polynomial roots $\lambda_1, \lambda_2, ..., \lambda_n,\ then$

$$AP = A\begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \dots & \vec{p}_n \end{bmatrix} = \begin{bmatrix} A\vec{p}_1 & A\vec{p}_2 & \dots & A\vec{p}_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\vec{p}_1 & \lambda_2\vec{p}_2 & \dots & \lambda_n\vec{p}_n \end{bmatrix}$$

$$= PD = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \dots & \vec{p}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

for eigenvalues λ_i of A with corresponding eigenvectors $\mathbf{p_i}$.

P is invertible iff the eigenvectors that form its columns are linearly independent iff $\dim(E_{\lambda_i}) = geometric\ multiplicity = a\text{lg}braic\ multiplicity\ for\ each\ distinct\ \lambda_i.$

A is Diagonalizable iff A is Similar to a Diagonal Matrix (Cont.)

This gives us n linearly independent eigenvectors for P, so P^{-1} exists. Therefore, A is diagonalizable since

$$P^{-1}AP = P^{-1}PD = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

The square matrices S and T are similar iff there exists a nonsingular P such that $S = P^{-1}TP$ or $PSP^{-1} = T$.

Since A is similar to a diagonal matrix, A is diagonalizable.

Another Example

Example 4: Solve the eigenvalue problem A**x** = λ**x** and find the eigenspace, algebraic multiplicity, and geometric multiplicity for each eigenvalue.

$$A = \begin{bmatrix} -4 & -3 & 6 \\ 0 & -1 & 0 \\ -3 & -3 & 5 \end{bmatrix}$$

Step 1: Write down the characteristic equation of A and solve for its eigenvalues.

$$p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda + 4 & 3 & -6 \\ 0 & \lambda + 1 & 0 \\ 3 & 3 & \lambda - 5 \end{vmatrix} = (-1)(\lambda + 1)\begin{vmatrix} \lambda + 4 & -6 \\ 3 & \lambda - 5 \end{vmatrix}$$

Another Example (Cont.)

$$
\begin{aligned}
& p(\lambda) \\
& = (\lambda+1)(\lambda+4)(\lambda-5)+18(\lambda+1)=0 \\
& = (\lambda^2+5\lambda+4)(\lambda-5)+18(\lambda+1)=0 \\
& = (\lambda^3+5\lambda^2+4\lambda-5\lambda^2-25\lambda-20)+18\lambda+18=0 \\
& = \lambda^3-3\lambda-2=(\lambda+1)(\lambda^2-\lambda-2)=(\lambda+1)(\lambda-2)(\lambda+1)=0 \\
& = (\lambda-2)(\lambda+1)^2=0.
\end{aligned}
$$

So the eigenvalues are $\lambda_1 = 2, \lambda_2 = -1.$

Since the factor (λ - 2) is first power, $\lambda_1 = 2$ is not a repeated root. $\lambda_1 = 2$ has an algebraic multiplicity of 1. On the other hand, the factor (λ +1) is squared, $\lambda_2 = -1$ is a repeated root, and it has an algebraic multiplicity of 2.

Another Example (Cont.)

Step 2: Use Gaussian elimination with back-substitution to solve $(\lambda I - A)\,\mathbf{x} = \mathbf{0}$ for λ_1 and λ_2.

For $\lambda_1 = 2$, the augmented matrix for the system is

$$\left[2I - A \mid \vec{0}\right] = \begin{bmatrix} 6 & 3 & -6 & 0 \\ 0 & 3 & 0 & 0 \\ 3 & 3 & -3 & 0 \end{bmatrix} \sim \begin{array}{c} \frac{1}{6}r1 \to r1 \\ \frac{1}{3}r2 \to r2 \\ r3 \end{array} \begin{bmatrix} 1 & 1/2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 3 & -3 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} r1 \\ r2 \\ -3r1 + r3 \to r3 \end{array} \begin{bmatrix} 1 & 1/2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \end{bmatrix}$$

$$: \begin{array}{c} r1 \\ r2 \\ -\frac{3}{2}r2 + r3 \to r3 \end{array} \begin{bmatrix} 1 & 1/2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case,

$x_3 = r$, $x_2 = 0$, and

$x_1 = -1/2(0) + r$

$= 0 + r = r.$

Another Example (Cont.)

Thus, the eigenvector corresponding to λ_1 = 2 is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, r \neq 0. \textit{ If we choose } \vec{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$\textit{then } B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \textit{is a basis for the eigenspace of } \lambda_1 = 2.$$

$$E_{\lambda_1} = span(\{\vec{p}_1\}) \textit{ and } \dim(E_{\lambda_1}) = 1, \textit{ so the geometric multiplicity is } 1.$$

$$A\vec{x} = 2\vec{x} \textit{ or } (2I - A)\vec{x} = \vec{0}.$$

$$\begin{bmatrix} -4 & -3 & 6 \\ 0 & -1 & 0 \\ -3 & -3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+6 \\ 0 \\ -3+5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Another Example (Cont.)

For $\lambda_2 = -1$, the augmented matrix for the system is

$$\left[(-1)I - A \mid \vec{0}\right] = \begin{bmatrix} 3 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & -6 & 0 \end{bmatrix} \sim \begin{array}{c} \frac{1}{3}r1 \to r1 \\ r2 \\ r3 \end{array} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & -6 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} r1 \\ r2 \\ -3r1 + r3 \to r3 \end{array} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3 = t$, $x_2 = s$, and $x_1 = -s + 2t$. Thus, the solution has two linearly independent eigenvectors for $\lambda_2 = -1$ with

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s + 2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, s \neq 0, t \neq 0.$$

Another Example (Cont.)

$$If\ \ we\ choose\ \vec{p}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\ and\ \vec{p}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},\ then\ B_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$is\ a\ basis\ \ for\ E_{\lambda_2} = span(\{\vec{p}_2, \vec{p}_3\})\ and\ \ \dim(E_{\lambda_2}) = 2,$$

$$so\ the\ geometric\ multiplicity\ is\ 2.$$

Another Example (Cont.)

Since the geometric multiplicity is equal to the algebraic multiplicity for each distinct eigenvalue, we found three linearly independent eigenvectors. The matrix A is diagonalizable since P = [$\mathbf{p}_1$ $\mathbf{p}_2$ $\mathbf{p}_3$] is nonsingular.

$$\textit{Thus, we have } AP = PD \textit{ as follows}:$$

$$\begin{bmatrix} -4 & -3 & 6 \\ 0 & -1 & 0 \\ -3 & -3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

Another Example (Cont.)

We can find P^{-1} as follows:

$$[P \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$: \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] : \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$: \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]. \quad So,\ P^{-1} = \begin{bmatrix} -1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Another Example (Cont.)

Note that A and D are similar matrices.

$$AP = PD \text{ gives us } A = APP^{-1} = PDP^{-1}.$$

$$Thus,\ PDP^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -4 & -3 & 6 \\ 0 & -1 & 0 \\ -3 & -3 & 5 \end{bmatrix} = A$$

Another Example (Cont.)

Also, D = P^{-1} AP =

$$D = \begin{bmatrix} -1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -4 & -3 & 6 \\ 0 & -1 & 0 \\ -3 & -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & 4 \\ 0 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

So, A and D are similar with D = P^{-1} AP and A = PD P^{-1} .

The Orthogonal Diagonalization of a Symmetric Matrix

If P is an orthogonal matrix, its inverse is its transpose, $P^{-1} = P^{T}$. Since

$$P^T P = \begin{bmatrix} \vec{p}_1^{\,T} \\ \vec{p}_2^{\,T} \\ \vec{p}_3^{\,T} \end{bmatrix} \begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 \end{bmatrix} == \begin{bmatrix} \vec{p}_1^{\,T}\vec{p}_1 & \vec{p}_1^{\,T}\vec{p}_2 & \vec{p}_1^{\,T}\vec{p}_3 \\ \vec{p}_2^{\,T}\vec{p}_1 & \vec{p}_2^{\,T}\vec{p}_2 & \vec{p}_2^{\,T}\vec{p}_3 \\ \vec{p}_3^{\,T}\vec{p}_1 & \vec{p}_3^{\,T}\vec{p}_2 & \vec{p}_3^{\,T}\vec{p}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{p}_1 \cdot \vec{p}_1 & \vec{p}_1 \cdot \vec{p}_2 & \vec{p}_1 \cdot \vec{p}_3 \\ \vec{p}_2 \cdot \vec{p}_1 & \vec{p}_2 \cdot \vec{p}_2 & \vec{p}_2 \cdot \vec{p}_3 \\ \vec{p}_3 \cdot \vec{p}_1 & \vec{p}_3 \cdot \vec{p}_2 & \vec{p}_3 \cdot \vec{p}_3 \end{bmatrix} = \begin{bmatrix} \vec{p}_i \cdot \vec{p}_j \end{bmatrix} = I \ \ because \ \ \vec{p}_i \cdot \vec{p}_j = 0$$

$$for \ \ i \neq j \ \ and \ \ \vec{p}_i \cdot \vec{p}_j = 1 \ \ for \ \ i = j = 1, 2, 3. \ So, \ \ P^{-1} = P^{T}.$$

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

A is a symmetric matrix if $A = A^T$. Let A be diagonalizable so that $A = PDP^{-1}$. But $A = A^T$ and

$$A^T = (PDP^{-1})^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A.$$

This shows that for a symmetric matrix A to be diagonalizable, P must be orthogonal.

If $P^{-1} \neq P^T$, then $A \neq A^T$. The eigenvectors of A are mutually orthogonal but not orthonormal. This means that the eigenvectors must be scaled to unit vectors so that P is orthogonal and composed of orthonormal columns.

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

Example 5: Determine if the symmetric matrix A is diagonalizable; if it is, then find the orthogonal matrix P that orthogonally diagonalizes the symmetric matrix A.

$$Let\ A = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix},\ then\ \det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & 1 & 0 \\ 1 & \lambda - 5 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix}$$

$$= (-1)^{3+1}(\lambda + 2)\begin{vmatrix} \lambda - 5 & 1 \\ 1 & \lambda - 5 \end{vmatrix} = (\lambda + 2)(\lambda - 5)^2 - (\lambda + 2)$$

$$= \lambda^3 - 8\lambda^2 + 4\lambda + 48 = (\lambda - 4)(\lambda^2 - 4\lambda - 12) = (\lambda - 4)(\lambda + 2)(\lambda - 6) = 0$$

$$Thus,\ \lambda_1 = 4,\ \lambda_2 = -2,\ \lambda_3 = 6.$$

Since we have three distinct eigenvalues, we will see that we are guaranteed to have three linearly independent eigenvectors.

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

Since $\lambda_1 = 4$, $\lambda_2 = -2$, and $\lambda_3 = 6$, are distinct eigenvalues, each of the eigenvalues has algebraic multiplicity 1.

An eigenvalue must have geometric multiplicity of at least one. Otherwise, we will have the trivial solution. Thus, we have three linearly independent eigenvectors.

We will use Gaussian elimination with back-substitution as follows:

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

$For\ \lambda_1 = 4\ ,$

$$\left[\lambda_1 I - A \middle| \vec{0}\right] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right]$$

$$: \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] : \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = s,\ \ x_3 = 0,\ \ x_1 = s\ .$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\ or\ \ \vec{p}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

$$For\ \ \lambda_2 = -2\ ,$$

$$\left[\lambda_2 I - A \middle| \vec{0}\right] = \left[\begin{array}{ccc|c} -7 & 1 & 0 & 0 \\ 1 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$: \left[\begin{array}{ccc|c} 1 & -7 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = s,\ \ x_2 = 0,\ \ x_1 = 0\ .$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\ or\ \ \vec{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

$$For\ \lambda_3 = 6,$$

$$\left[\lambda_3 I - A \,\middle|\, \vec{0}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 8 & 0 \end{array}\right] : \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_2 = s,\ \ x_3 = 0,\ \ x_1 = -s.$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\ or\ \ \vec{p}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

As we can see the eigenvectors of A are distinct, so {$\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$} is linearly independent, P^{-1} exists for P =[$\mathbf{p}_1$ $\mathbf{p}_2$ $\mathbf{p}_3$] and

$AP = PD \Leftrightarrow PDP^{-1}$. Thus A is diagonalizable.

Since A = A^T (A is a symmetric matrix) and P is orthogonal with approximate scaling of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $P^{-1} = P^T$.

$$PP^{-1} = PP^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

As we can see the eigenvectors of A are distinct, so {$\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$} is linearly independent, P^{-1} exists for P =[$\mathbf{p}_1$ $\mathbf{p}_2$ $\mathbf{p}_3$] and

$AP = PD \Leftrightarrow PDP^{-1}.$ Thus A is diagonalizable.

Since A = A^T (A is a symmetric matrix) and P is orthogonal with approximate scaling of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $P^{-1} = P^T$.

$$PP^{-1} = PP^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

Note that A and D are similar matrices. PD P^{-1} =

$$PDP^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4/\sqrt{2} & 0 & -6/\sqrt{2} \\ 4/\sqrt{2} & 0 & 6/\sqrt{2} \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix} = A.$$

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

Also, D = $P^{-1}AP = P^TAP$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4/\sqrt{2} & 4/\sqrt{2} & 0 \\ 0 & 0 & -2 \\ -6/\sqrt{2} & 6/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

So, A and D are similar with D = P^TAP and A = PDP^T.

The Orthogonal Diagonalization of a Symmetric Matrix (Cont.)

$A^T = (PD\,P^{-1})^T = (PD\,P^T)^T = (P^T)^T D^T P^T = P\,D^T P^T$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}^T \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4/\sqrt{2} & 0 & -6/\sqrt{2} \\ 4/\sqrt{2} & 0 & 6/\sqrt{2} \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

= A. This shows that if A is a symmetric matrix, P must be orthogonal with $P^{-1} = P^T$.

How to Determine the Power and the Eigenvalues of a Matrix, A^k?

From Example 1, the diagonal matrix for matrix A is:

$$D=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}=\begin{bmatrix}-5 & 0\\ 0 & 2\end{bmatrix}=P^{-1}AP=D\Rightarrow A=PDP^{-1},$$

$$\text{and } A^3=PDP^{-1}PDP^{-1}PDP^{-1}=PD^3P^{-1}$$

$$=\begin{bmatrix}1 & -2\\ 1 & 5\end{bmatrix}\begin{bmatrix}(-5)^3 & 0\\ 0 & (2)^3\end{bmatrix}\begin{bmatrix}5/7 & 2/7\\ -1/7 & 1/7\end{bmatrix}=\begin{bmatrix}125 & -16\\ 125 & 40\end{bmatrix}\begin{bmatrix}5/7 & 2/7\\ -1/7 & 1/7\end{bmatrix}$$

$$=\begin{bmatrix}641/7 & 234/7\\ 585/7 & 290/7\end{bmatrix}.\ \text{For } A^3,\text{the eigenvalues, are } \lambda_1^3=-125 \text{ aand } \lambda_2^3=8.$$

In general, the power of a matrix, $A^k=PD^kP^{-1}$. *and the eigenvalues are* λ_i^k, *where* λ_i *is on the main diagonal of* D.

What have we learned?

We have learned to:

1. Solve the eigenvalue problem by finding the eigenvalues and the corresponding eigenvectors of an n x n matrix. Find the algebraic multiplicity and the geometric multiplicity of an eigenvalue.
2. Find a basis for each eigenspace of an eigenvalue.
3. Determine whether a matrix A is diagonalizable.
4. Find a matrix P, P^{-1}, and D that diagonalize A if A is diagonalizable.
5. Find an orthogonal matrix P with $P^{-1} = P^T$ and D that diagonalize A if A is symmetric and diagonalizable.
6. Determine the power and the eigenvalues of a matrix, A^k.

Credit

Some of these slides have been adapted/modified in part/whole from the following textbook:

- Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition