Chapter 5
Determinants

5.1 Introduction

Every square matrix has associated with it a scalar called its determinant.

Given a matrix **A**, we use **det(A)** or **|A|** to designate its determinant.

We can also designate the determinant of matrix **A** by replacing the brackets by vertical straight lines. For example,

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \qquad \det(A) = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}$$

Definition 1: The determinant of a 1×1 matrix [a] is the scalar a.

Definition 2: The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the scalar ad-bc.

For higher order matrices, we will use a recursive procedure to compute determinants.

5.2 Expansion by Cofactors

Definition 1: Given a matrix A, a minor is the determinant of any square submatrix of A.

Definition 2: Given a matrix $\mathbf{A=[a_{ij}]}$, the **cofactor** of the element $\mathbf{a_{ij}}$ is a scalar obtained by multiplying together the term $\mathbf{(-1)^{i+j}}$ and the minor obtained from **A** by removing the ***i***th row and the ***j***th column.

In other words, the cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j}M_{ij}$.

For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \Rightarrow C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

5.2 Expansion by Cofactors

To find the determinant of a matrix A of arbitrary order,

Pick any one row or any one column of the matrix;

For each element in the row or column chosen, find its cofactor;

Multiply each element in the row or column chosen by its cofactor and sum the results. This sum is the determinant of the matrix.

In other words, the determinant of **A** is given by

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

*i*th row expansion

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

*j*th column expansion

Example 1:
We can compute the determinant

$$|\mathbf{T}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

by expanding along the first row,

$$|\mathbf{T}| = 1 \times (-)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2 \times (-)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \times (-)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \quad = -3 + 12 - 9 = 0$$

Or expand down the second column:

$$|\mathbf{T}| = 2 \times (-)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \times (-)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8 \times (-)^{3+2} \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \quad = 12 - 60 + 48 = 0$$

Example 2: (using a row or column with many zeroes)

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = 1 \times (-)^{2+3} \begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix} \quad = 16$$

5.3 Properties of determinants

Property 1: If one row of a matrix consists entirely of zeros, then the determinant is zero.

Property 2: If two rows of a matrix are interchanged, the determinant changes sign.

Property 3: If two rows of a matrix are identical, the determinant is zero.

Property 4: If the matrix **B** is obtained from the matrix **A** by multiplying every element in one row of **A** by the scalar λ, then $|\mathbf{B}| = \lambda|\mathbf{A}|$.

Property 5: For an $n \times n$ matrix **A** and any scalar λ, $\det(\lambda\mathbf{A}) = \lambda^n \det(\mathbf{A})$.

5.3 Properties of determinants

Property 6: If a matrix **B** is obtained from a matrix **A** by adding to one row of **A**, a scalar times another row of **A**, then $|\mathbf{A}|=|\mathbf{B}|$.

Property 7: $\det(\mathbf{A}) = \det(\mathbf{A}^T)$.

Property 8: The determinant of a triangular matrix, either upper or lower, is the product of the elements on the main diagonal.

Property 9: If A and B are of the same order, then

$$\det(AB)=\det(A)\det(B).$$

5.4 Pivotal condensation

Properties 2, 4, 6 of the previous section describe the effects on the determinant when applying row operations.

These properties comprise part of an *efficient algorithm* for computing determinants, technique known as pivotal condensation.

-A given matrix is transformed into row-reduced form using elementary row operations

-A record is kept of the changes to the determinant as a result of properties 2, 4, 6.

-Once the transformation is complete, the row-reduced matrix is in upper triangular form, and its determinant is easily found by property 8.

Example in the next slide

5.4 Pivotal condensation

- Find the determinant of

$$A=\begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix} \times(-2) = -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}$$

Factor –7 out of the 2nd row

$$= 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} \times(-1) = 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = 7(1)(1)(-1) = -7$$

5.5 Inversion

Theorem 1: A square matrix has an inverse if and only if its determinant is not zero.

Below we develop a method to calculate the inverse of nonsingular matrices using determinants.

Definition 1: The **cofactor matrix** associated with an $n \times n$ matrix $\mathbf{A}$ is an $n \times n$ matrix $\mathbf{A}^c$ obtained from $\mathbf{A}$ by replacing each element of $\mathbf{A}$ by its cofactor.

Definition 2: The **adjugate** of an $n \times n$ matrix $\mathbf{A}$ is the transpose of the cofactor matrix of $\mathbf{A}$: $\mathbf{A}^a = (\mathbf{A}^c)^T$

Example of finding adjugate

- Find the adjugate of
$$A=\begin{bmatrix}-1 & 3 & 2\\ 0 & -2 & 1\\ 1 & 0 & -2\end{bmatrix}$$

Solution:

The cofactor matrix of *A*:

$$\begin{bmatrix}\begin{vmatrix}-2 & 1\\ 0 & -2\end{vmatrix} & -\begin{vmatrix}0 & 1\\ 1 & -2\end{vmatrix} & \begin{vmatrix}0 & -2\\ 1 & 0\end{vmatrix}\\ -\begin{vmatrix}3 & 2\\ 0 & -2\end{vmatrix} & \begin{vmatrix}-1 & 2\\ 1 & -2\end{vmatrix} & -\begin{vmatrix}-1 & 3\\ 1 & 0\end{vmatrix}\\ \begin{vmatrix}3 & 2\\ -2 & 1\end{vmatrix} & -\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix} & \begin{vmatrix}-1 & 3\\ 0 & -2\end{vmatrix}\end{bmatrix}=\begin{bmatrix}4 & 1 & 2\\ 6 & 0 & 3\\ 7 & 1 & 2\end{bmatrix}$$

$$A^a=\begin{bmatrix}4 & 6 & 7\\ 1 & 0 & 1\\ 2 & 3 & 2\end{bmatrix}$$

Inversion using determinants

Theorem 2: **A Aᵃ = Aᵃ A = |A| I .**

If **|A| ≠ 0** then from Theorem 2,

$$A\left(\frac{A^a}{|A|}\right)=\left(\frac{A^a}{|A|}\right)A=I$$

$$A^{-1}=\frac{1}{|A|}A^a \quad if \quad |A|\neq 0$$

That is, if |A| ≠ 0, then A^{-1} may be obtained by dividing the adjugate of A by the determinant of A.

For example, if $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix},$

then

$$A^{-1}=\frac{1}{|A|}A^a=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inversion using determinants: example

Use the adjugate of $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ to find A^{-1}

$$|A| = (-1)(-2)(-2) + (3)(1)(1) - (1)(-2)(2) = 3$$

$$\mathrm{A}^{a} = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|\mathrm{A}|} A^{a} = \frac{1}{3} \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

5.6 Cramer's rule

- If a system of *n* linear equations in *n* variables **Ax=b** has a coefficient matrix with a *nonzero* determinant |A|, then the solution of the system is given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

where $\boldsymbol{A_i}$ is a matrix obtained from **A** by replacing the ***i***th column of **A** by the vector **b**.

- *Example*:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

$$x_3 = \frac{|A_3|}{|A|} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} \Bigg/ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

5.6 Cramer's rule: example

- Use Cramer's Rule to solve the system of linear equation.

$$
\begin{aligned}
-x \quad + 2y \quad - 3z &= 1 \\
2x \qquad\qquad + \ z &= 0 \\
3x \quad - 4y \quad + 4z &= 2
\end{aligned}
$$

$$
|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10
$$

$$
x = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{(2)(1)(2) - (-4)(1)(1)}{10} = \frac{8}{10} = \frac{4}{5}
$$

$$
y = -\frac{3}{2}, \quad z = -\frac{8}{5}
$$