Complex Numbers in Fractal Visual Arts

Annisaur Rosi Lutfiana 13512088[1]
Program Studi Informatika
Sekolah Teknik Elektro dan Informatika
Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia
[1]*annisaurrosi@s.itb.ac.id*

***Abstract*—The applications of geometric algebra are extensive and knowledgeable in varieties of disciplines, both in theoretical and applied sciences and technologies. Another part of the geometric algebra that can be widely applied is complex numbers. The use of complex numbers is indeed very useful to symbolize numbers that considered as unnatural and to represents mathematical things that built in non-integer dimension.**

The following is a brief explanation of the application of complex numbers in the form of fractal visualizations, a new branch between mathematics and visual art. Later on, fractals offer almost unlimited ways of describing, measuring and predicting natural phenomena.

***Keywords*—Complex numbers, fractal, Mandelbrot set, Julia set.**

I. Introduction

Most physical systems of nature and many human artifacts in this world cannot be represented by regular geometric shapes of the standard geometry derived from Euclid. Geometric algebra offers almost unlimited ways of describing, measuring and predicting these natural phenomena. In this era of rapid sciences and technologies developments, people try to make possible to define the whole world using mathematical equations.

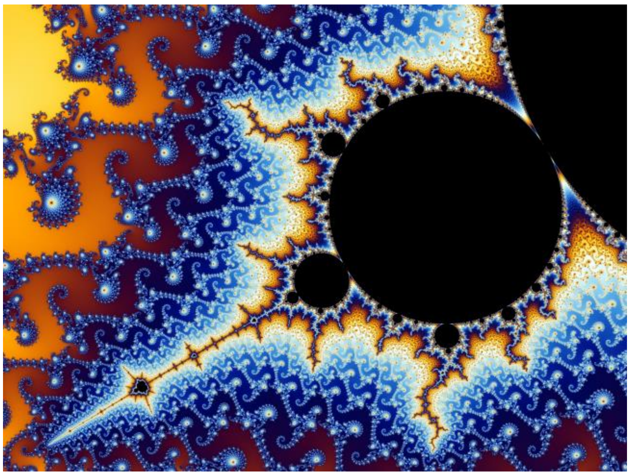

Figure 1: An example of fractal visualization. (Source: Wikipedia.org)

Many people are fascinated by the beautiful images termed fractals. Fractals is a new branch of mathematics and visual art. Perhaps this is the reason why most people recognize fractals only as pretty pictures useful as backgrounds on the computer screen or original postcard patterns. In fact, only few people know that some fractals are made by deep comprehension in mathematics and understand the complexity of algorithms behind them.

Extending beyond the typical perception of mathematics as a body of complicated, boring formulas, fractal geometry mixes visual art with mathematics to demonstrate that equations are more than just a collection of numbers. What makes fractals even more interesting is that they are the best existing mathematical descriptions of many natural forms, such as coastlines, mountains or parts of living organisms.

Two of the most important properties of fractals are self-similarity and non-integer dimension.

Self-similarity can be explained in following condition. If we look carefully at a fern leaf, we will notice that every little leaf - part of the bigger one - has the same shape as the whole fern leaf. We can say that the fern leaf is self-similar. The same is with fractals: we can magnify them many times and after every step we will see the same shape, which is characteristic of that particular fractal.

The non-integer dimension is rather more difficult to explain. Classical geometry deals with objects of integer dimensions: zero dimensional points, one dimensional lines and curves, two dimensional plane figures such as squares and circles, and three dimensional solids such as cubes and spheres. However, many natural phenomena are better described using a dimension between two whole numbers. So while a straight line has a dimension of one, a fractal curve will have a dimension between one and two, depending on how much space it takes up as it twists and curves. The more the flat fractal fills a plane, the closer it approaches two dimensions. Likewise, a "hilly fractal scene" will reach a dimension somewhere between two and three. So a fractal landscape made up of a large hill covered with tiny mounds would be close to the second dimension, while a rough surface composed of many medium-sized hills would be close to the third dimension.

The mathematics behind fractals are incredibly interesting and captivating. We need to have a grasp on algebra and some complex number background is preferable. We already described how fractals are created through applying functions, but never explained any functions and how they work. In this section, we will describe the two most popular fractal sets and how they work, the Julia set and Mandelbrot set.

II. RELATED THEORIES

2.1 Complex Numbers

A complex number is a number that can be expressed in the form a + bi, where a and b are real numbers and i is the imaginary unit, that satisfies the equation $i^2 = -1$.[1] In this expression, a is the real part and b is the imaginary part of the complex number.

The real number *a* is called the *real part* of the complex number *a* + *bi*; the real number *b* is called the *imaginary part* of *a* + *bi*. By this convention the imaginary part does not include the imaginary unit: hence *b*, not *bi*, is the imaginary part.[2][3] The real part of a complex number *z* is denoted by Re(*z*) or $\Re(z)$; the imaginary part of a complex number *z* is denoted by Im(*z*) or $\Im(z)$. For example,

$$\operatorname{Re}(-3.5 + 2i) = -3.5$$
$$\operatorname{Im}(-3.5 + 2i) = 2.$$

Hence, in terms of its real and imaginary parts, a complex number *z* is equal to Re(z) + Im(z).i. This expression is sometimes known as the Cartesian form of *z*.

A real number *a* can be regarded as a complex number *a* + 0*i* whose imaginary part is 0. A purely imaginary number *bi* is a complex number 0 + *bi* whose real part is zero. It is common to write *a* for *a* + 0*i* and *bi* for 0 + *bi*. Moreover, when the imaginary part is negative, it is common to write *a* − *bi* with *b* > 0 instead of *a* + (−*b*)*i*, for example 3 − 4*i* instead of 3 + (−4)*i*. The set of all complex numbers is denoted by $\mathbb{C}$.

Complex numbers extend the concept of the one-dimensional number line to the two-dimensional complex plane by using the horizontal axis for the real part and the vertical axis for the imaginary part. The complex number a + bi can be identified with the point (a, b) in the complex plane. A complex number whose real part is zero is said to be purely imaginary, whereas a complex number whose imaginary part is zero is a real number. In this way, the complex numbers contain the ordinary real numbers while extending them in order to solve problems that cannot be solved with real numbers alone.

2.2 Julia (and Fatou) Set

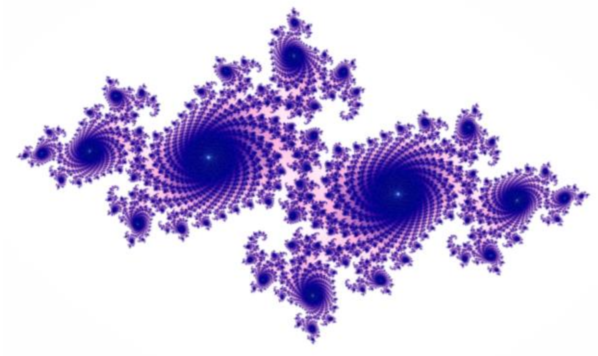

Figure 2: An example of fractal generated from Julia set. (Source: Wikipedia.org)

In the context of complex dynamics, a topic of mathematics, the Julia set and the Fatou set are two complementary sets (Julia 'laces' and Fatou 'dusts') defined from a function. Informally, the Fatou set of the function consists of values with the property that all nearby values behave similarly under repeated iteration of the function, and the Julia set consists of values such that an arbitrarily small perturbation can cause drastic changes in the sequence of iterated function values. Thus the behavior of the function on the Fatou set is 'regular', while on the Julia set its behavior is 'chaotic'.

The Julia set of a function *f* is commonly denoted *J*(*f*), and the Fatou set is denoted *F*(*f*).[4] The complement of *F*(*f*) is the Julia set *J*(*f*) of *f*(*z*). *J*(*f*) is a nowhere dense set (it is without interior points) and an uncountable set (of the same cardinality as the real numbers). Like *F*(*f*), *J*(*f*) is left invariant by *f*(*z*), and on this set the iteration is repelling, meaning that

$$|f(z) - f(w)| > |z - w|$$

for all *w* in a neighbourhood of *z* (within *J*(*f*)). This means that *f*(*z*) behaves chaotically on the Julia set. Although there are points in the Julia set whose sequence of iterations is finite, there are only a countable number of such points (and they make up an infinitely small part of the Julia set). The sequences generated by points outside this set behave chaotically, a phenomenon called *deterministic chaos*.

The Julia set and the Fatou set of *f* are both completely invariant under iterations of the holomorphic function *f*.

$$f^{-1}(J(f)) = f(J(f)) = J(f)$$
$$f^{-1}(F(f)) = f(F(f)) = F(f)$$

The Julia Set iterates the equation $z_{n+1} = z_n^2 + z_c$ where $z_0 = 0$. The number z is a complex number. Then it plots pixels for different values of z_c, we assume z_c is a given constant for all the pixels and we plot different values of z_0. Therefore, there are infinite Julia Sets depending on what value chosen for z_c.

For example, let's say we want to color a pixel at (-1,0.5) for the Julia Set using the constant z_c = -1.125+0.25i. We start out with z_0 = -1+0.5i:

$z_1 = z_0^2+z_c = -0.375-0.75i$

$z_2 = z_1^2+z_c = -1.54688-0.8125i$

$z_3 = z_2^2+z_c = -0.60767+2.26367i$

We stop here because the magnitude $|z_3|$ = 2.34381 > 2. We conclude that this point is unbounded and color code our pixel based on n = 3.

2.3 Mandelbrot Set

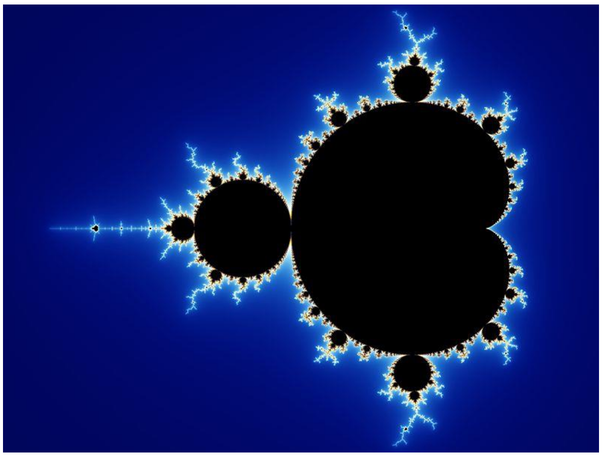

Figure 3: An example of fractal generated from Mandelbrot set. (Source: Wikipedia.org)

The Mandelbrot set is the set of complex numbers *c* for which the sequence (c, $c^2 + c$, $(c^2+c)^2 + c$, $((c^2+c)^2+c)^2 + c$, $(((c^2+c)^2+c)^2+c)^2 + c$, …) does not approach infinity. The set is closely related to Julia sets (which include similarly complex shapes).

Mandelbrot set images are made by sampling complex numbers and determining for each whether the result tends towards infinity when a particular mathematical operation is iterated on it. Treating the real and imaginary parts of each number as image coordinates, pixels are colored according to how rapidly the sequence diverges, if at all.

More precisely, the Mandelbrot set is the set of values of *c* in the complex plane for which the orbit of 0 under iteration of the complex quadratic polynomial

$$z_{n+1} = z_n^2 + c$$

remains bounded.[5] That is, a complex number *c* is part of the Mandelbrot set if, when starting with $z_0 = 0$ and applying the iteration repeatedly, the absolute value of z_n remains bounded however large *n* gets. This can also be represented as

$$z_{n+1} = z_n^2 + c$$
$$c \in M \iff \lim_{n\to\infty} |z_{n+1}| \leq 2$$

For example, letting *c* = 1 gives the sequence 0, 1, 2, 5, 26,…, which tends to infinity. As this sequence is unbounded, 1 is not an element of the Mandelbrot set. On the other hand, *c* = −1 gives the sequence 0, −1, 0, −1, 0,…, which is bounded, and so −1 belongs to the Mandelbrot set.

In short, the difference between Mandelbrot and Julia set is that for the same algorithm used in Julia set in Mandelbrot we define the set of all complex numbers z_c such that z_n is finite as n goes to infinity. There is a theorem which states that it will not be finite if the magnitude of z_n ever exceeds 2.

In order to generate an image, we must iterate this equation repeatedly for every value of z_c in the complex plane. For example, let's say we want to color a pixel at (-1,0.5). We start out with z_c = -1+0.5i and z_0 = 0 and proceed from there:

$z_1 = z_0^2+z_c = -1+0.5i$

$z_2 = z_1^2+z_c = -0.25-0.5i$

$z_3 = z_2^2+z_c = -1.1875+0.75i$

$z_4 = z_3^2+z_c = -0.1529344-1.28125i$

$z_5 = z_4^2+z_c = -2.61839+0.890381i$

We stop here because the magnitude $|z_5|$ = 2.76564 > 2. We conclude that this point is unbounded and color code our pixel based on the required number of iterations n = 5. Then we repeat the whole process again for each pixel.

Images of the Mandelbrot set display an elaborate boundary that reveals progressively ever-finer recursive detail at increasing magnifications. The "style" of this repeating detail depends on the region of the set being examined. The set's boundary also incorporates smaller versions of the main shape, so the fractal property of self-similarity applies to the entire set, and not just to its parts.

The Mandelbrot set has become popular outside mathematics both for its aesthetic appeal and as an example of a complex structure arising from the application of simple rules, and is one of the best-known examples of mathematical visualization.

III. Implementation and Experiments

3.1 Julia Set

- For example, c = (-0.7,0.27015), -0.7 is the real part and 0.27015 the imaginary part.
- Transform the coordinates so it lies between -1 and 1 so the coordinates become (1,0.27015) and p = 1 + 0.27015i.
- Apply the function for the first time:

```
z = p² + c
    = (1 + 0. 27015i)² - 0.7 + 0. 27015i
    = 1 + 2*0. 27015i + 0.07890*i² - 0.7
```

```
+ 0. 27015i
  = 0.3 + 0.81045i - 0.07890
  = 0.2211 + 0.81045i
```

- So z = (0.2211,0.81045), and the distance of z to the origin = sqrt(0.2211*0.2211 + 0.81045*0.81045) = 0.84. It will make a circle inside a plane with radius of 2.
- Continue the iteration until it gets z = 2. The number of iterations represents the number of colors.

The algorithm for drawing the Julia set fractals is represented below.

```
int main(int argc, char *argv[])
{
    screen(400, 300, 0, "Julia Set");
    double cRe, cIm;
    double newRe, newIm, oldRe, oldIm;
    double zoom = 1, moveX = 0, moveY = 0;
    ColorRGB color;
    int maxIterations = 300;
    cRe = -0.7;
    cIm = 0.27015;

    //loop through every pixel
    for(int x = 0; x < w; x++)
    for(int y = 0; y < h; y++)
    {
        newRe = 1.5 * (x - w / 2) / (0.5 *
zoom * w) + moveX;
        newIm = (y - h / 2) / (0.5 * zoom *
h) + moveY;
        int i;
        for(i = 0; i < maxIterations; i++)
        {
            oldRe = newRe;
            oldIm = newIm;
            newRe = oldRe * oldRe - oldIm *
oldIm + cRe;
            newIm = 2 * oldRe * oldIm +
cIm;
            //if the point is outside the
circle with radius 2: stop
            if((newRe * newRe + newIm *
newIm) > 4) break;
        }
        color = HSVtoRGB(ColorHSV(i % 256,
255, 255 * (i < maxIterations)));
        //draw the pixel
        pset(x, y, color);
    }
    redraw();
    sleep();
    return 0;
}
```

The generated fractal shown below.

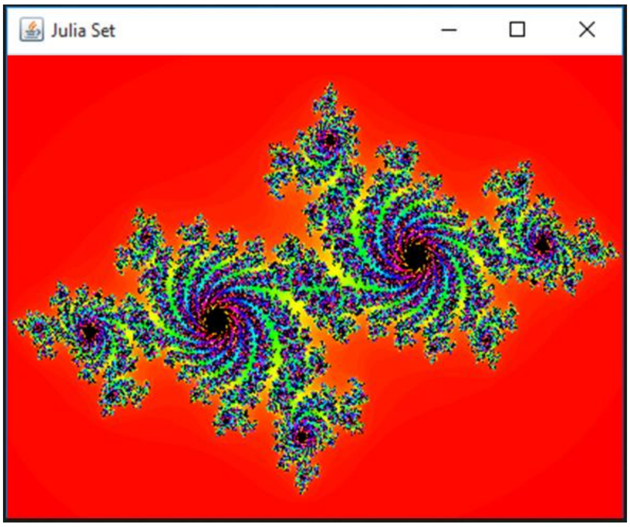

Figure 4: Output of Java program using Julia set algorithm.

3.2 Mandelbrot Set

To generate a Mandelbrot set we use the same iterative function as the Julia Set, only this time c will represent the position of the pixel, and z will start at (0,0).

The algorithm for drawing the Mandelbrot set fractals is represented below.

```
int main(int argc, char *argv[])
{
    screen(400, 300, 0, "Mandelbrot Set");
    double pr, pi;
    double newRe, newIm, oldRe, oldIm;
    double zoom = 1, moveX = -0.5, moveY =
0;
    ColorRGB color;
    int maxIterations = 300;

    //loop through every pixel
    for(int x = 0; x < w; x++)
    for(int y = 0; y < h; y++)
    {
         pr = 1.5 * (x - w / 2) / (0.5 *
zoom * w) + moveX;
        pi = (y - h / 2) / (0.5 * zoom * h)
+ moveY;
        newRe = newIm = oldRe = oldIm = 0;
        int i;
        //start the iteration process
        for(i = 0; i < maxIterations; i++)
        {
            oldRe = newRe;
            oldIm = newIm;
            newRe = oldRe * oldRe - oldIm *
oldIm + pr;
            newIm = 2 * oldRe * oldIm + pi;
           if((newRe * newRe + newIm *
newIm) > 4) break;
        }
        color = HSVtoRGB(ColorHSV(i % 256,
255, 255 * (i < maxIterations)));
         //draw the pixel
         pset(x, y, color);
    }
    redraw();
    sleep();
    return 0;
}
```

The generated fractal shown below.

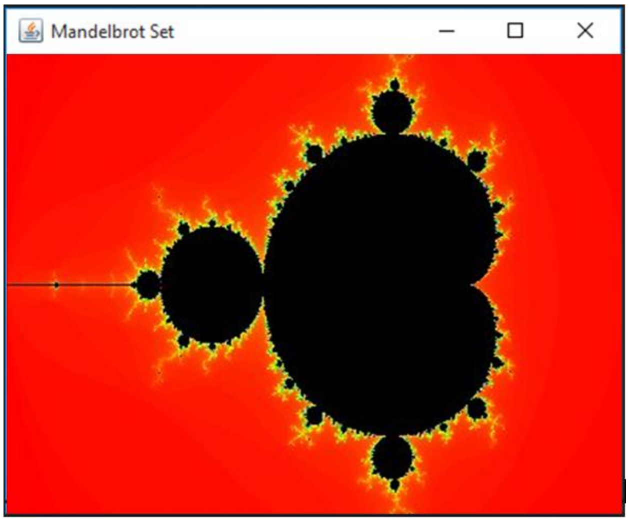

Figure 5: Output of Java program using Mandelbrot set algorithm.

IV. Conclusions

From all the explanations that has been described, the use of algebraic geometry in general and complex numbers in particular can be applied to various fields. Fractal art as one of the fields of visual art proved that algorithms and complex mathematical equations can even produce something beautiful with aesthetic value.

Julia sets and Mandelbrot sets are two of so many types of fractals that can demonstrate a decent and quite clear association between the application of mathematical algorithms with complex numbers and its visualization. Both of the sets use an algorithm that has similar terms of iterations yet able to generate very distinctive visualization, due to their representation of different constants. By using these algorithms, even any of complex fractal generation that can be found is likely to be explained and understandable.

VI. Acknowledgment

Annisaur Rosi Lutfiana, as the author of this paper, want to express her grateful thanks to God for His blessing during the making of this paper, her deepest gratitude to Dr. Ir. Rinaldi Munir, M.T. and Drs. Judhi Santoso, M.Sc. as the lecturers of IF2123–Aljabar Geometri.
Special thanks to my family back home and all my friends in HMIF for all the supports and guidances.
Also thanks for all of the authors which I got a lot of this study references from, may God bless your knowledge with endless abundances.

References

[1] Charles P. McKeague (2011), *Elementary Algebra*, Brooks/Cole, p. 524.
[2] Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA).
[3] Aufmann, Richard N.; Barker, Vernon C.; Nation, Richard D. (2007), "Chapter P", *College Algebra and Trigonometry* (6 ed.), Cengage Learning, p. 66.
[4] Gaston Julia (1918) "Mémoire sur l'iteration des fonctions rationnelles," *Journal de Mathématiques Pures et Appliquées*, vol. 8, pages 47–245.
[5] *"Mandelbrot Set Explorer: Mathematical Glossary"*. Retrieved 2007-10-07.

Statement

I hereby stated that this paper is copyrighted to myself, neither a copy from other‘s paper nor a translation of similar paper.

Bandung, 10th of December, 2015

Annisaur Rosi Lutfiana 13512088