# Quaternions

Quaternions are expressions of the form $a + bi + cj + dk$, where $a$, $b$, $c$, and $d$ are ordinary real numbers. For example, $3 + 2i - 5j + 4k$ and $\sqrt{2} - \frac{1}{2}i + \frac{2}{3}j$ are quaternions.

This number system is like the real numbers or the complex numbers in that you can add, subtract, multiply, and divide (as long as you don't divide by 0). We're going to learn how these operations work for quaternions.

Addition (or subtraction) is easy: just add (or subtract) each part separately. For example,

$$\begin{aligned}
&(3 + 2i - 5j + 4k) + (4 - 5i - 3j - k) = \\
&(3 + 4) + (2i - 5i) + (-5j - 3j) + (4k - k) = \\
&7 - 3i - 8j + 3k.
\end{aligned}$$

**Problem 1.** $(2 + 3i + j - k) + (4 + 5i - 2j + 6k)$

**Problem 2.** $(3 + 3i + 2j + 2k) - (6 - 4i + 3j + 5k)$

**Problem 3.** $(-2 - \frac{1}{2}i - 2j + \frac{2}{5}k) + (\frac{1}{3} + 2i + \frac{1}{4}j + k)$

Multiplication of quaternions is the tricky part. It's similar in spirit to multiplication of complex numbers, but it's a lot more complicated. You have to use FOIL (that is, the distributive law), but there are many terms that you have to keep track of.

You also need to know the following rules:

$$\begin{aligned}
i^2 &= j^2 = k^2 = -1 \\
ij &= k \qquad ji = -k \\
jk &= i \qquad kj = -i \\
ki &= j \qquad ik = -j.
\end{aligned}$$

You might notice something surprising and subtle about the above formulas. **The order in which you multiply quaternions matters!** For example, $i$ times $j$ is not equal to $j$ times $i$. This is a very interesting and important property of quaternions, even though it makes computations harder to do.

Here's an example of a multiplication problem for quaternions:

$$\begin{aligned}
& (3+2i+5j+4k)(4+5i+3j+k) \\
= \;& 3\cdot 4+3\cdot 5i+3\cdot 3j+3\cdot k+2i\cdot 4+2i\cdot 5i+2i\cdot 3j+2i\cdot k+ \\
& 5j\cdot 4+5j\cdot 5i+5j\cdot 3j+5j\cdot k+4k\cdot 4+4k\cdot 5i+4k\cdot 3j+4k\cdot k \\
= \;& 12+15i+9j+3k+8i+10i^2+6ij+2ik+ \\
& 20j+25ji+15j^2+5jk+16k+20ki+12kj+4k^2 \\
= \;& 12+15i+9j+3k+8i-10+6k-2j+ \\
& 20j-25k-15+5i+16k+20j-12i-4 \\
= \;& (12-10-15-4)+(15i+8i+5i-12i)+ \\
& (9j-2j+20j+20j)+(3k+6k-25k+16k) \\
= \;& -17+16i+47j+0k.
\end{aligned}$$

**Problem 4.** $(3+3i+5j+2k)(6+4i+j+k)$

**Problem 5.** $(8-2i+3j-k)(8+2i-3j+k)$

Problem 5 shows an important property of quaternions. If you multiply $a+bi+cj+dk$ with its "conjugate" $a-bi-cj-dk$, then you will always obtain an ordinary real number as the answer. This is important for division of quaternions.

We're just going to look at a special case of division for quaternions. We're only going to learn how to compute reciprocals of the form

$$\frac{1}{a+bi+cj+dk}.$$

The idea is the same as for complex numbers: multiply and divide by the conjugate. For example,

$$\begin{aligned}
\frac{1}{1+2i+3j+4k} &= \frac{1-2i-3j-4k}{(1+2i+3j+4k)(1-2i-3j-4k)} \\
&= \frac{1-2i-3j-4k}{1^2+2^2+3^2+4^2} \\
&= \frac{1-2i-3j-4k}{30} \\
&= \frac{1}{30}-\frac{2}{30}i-\frac{3}{30}j-\frac{4}{30}k \\
&= \frac{1}{30}-\frac{1}{15}i-\frac{1}{10}j-\frac{2}{15}k.
\end{aligned}$$

**Problem 6.** $\dfrac{1}{8-2i+3j-k}$.

**Problem 7.** $\dfrac{1}{-i+3j-5k}$.

Once you know how to take reciprocals, you can also learn how to divide quaternions, but we're not going to worry about that today.

**Bonus Problem.** Suppose that $b$, $c$, and $d$ are real numbers such that $b^2+c^2+d^2=1$. Show that $(bi+cj+dk)^2=-1$. Give a geometric description of these "square roots of $-1$".

Now you know of three different number systems in which you can add, subtract, multiply, and divide (except for division by 0). These three systems are the real numbers, the complex numbers, and the quaternions.

Are there any other similar number systems? For example, can you add, subtract, multiply, and divide expressions of the form $a+bi+cj$? It turns out to be easy to find number systems in which you can add, subtract, and multiply. But it is much harder to find a number system in which you can also divide. This is a very difficult problem whose solution was found in the 1950's.

In fact, there are only four possible number systems that have division. Three are the ones you already know about. The fourth number system is called the "octonions".

There are at least two places outside of mathematics where quaternions are useful. First, they are used in computer graphics to describe 3-dimensional rotations. Second, they show up in physics when taking cross products of 3-dimensional vectors.

For more general information about quaternions, here are a few websites:

`http://mathworld.wolfram.com/Quaternion.html`
`http://en.wikipedia.org/wiki/Quaternion`

For some colorful history on how the quaternions were invented, see:

`http://www.theworld.com/~sweetser/quaternions/intro/history/history.html`

A Google search for "quaternions" will turn up lots of other websites.